Homework 5 for 506, Spring 2009
due Friday, May 15

Problem 1.[10pt] Show that there is a canonical isomorphism $A/\mathfrak{a} \otimes_A M \simeq M/\mathfrak{a}M$.

Problem 2.[10pt] Let A be a local ring, and M, N be finitely generated A-modules. Show that if $M \otimes_A N = 0$ then either $M = 0$ or $N = 0$.

Problem 3.[10pt] Let M be a flat A-module, and B be an A-algebra. Show that $B \otimes_A M$ is a flat B-module.

Problem 4.[30pt] Let M be an A-module. The **support** of M, denoted $\operatorname{Supp} M$, is a subset of $\operatorname{Spec} A$ defined as follows:

$$\operatorname{Supp} M = \{\mathfrak{p} \in A \mid M_{\mathfrak{p}} \neq 0\}.$$

Prove the following properties of supports:

1. (5pt) $M \neq 0 \Leftrightarrow \operatorname{Supp} M \neq \emptyset$.
2. (5pt) For an ideal $\mathfrak{a} \in A$, $V(\mathfrak{a}) = \operatorname{Supp} A/\mathfrak{a}$.
3. (5pt) For any short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, we have $\operatorname{Supp} M \subset \operatorname{Supp} M' \cup \operatorname{Supp} M''$.
4. (5pt) $\operatorname{Supp} (\bigoplus M_i) = \bigcup \operatorname{Supp} M_i$
5. (5pt) If M is a finitely generated A-module, then $\operatorname{Supp} M = V(\mathrm{ann(M)})$ (Here, $\mathrm{ann(M)} = \{\mathrm{a \in A \mid aM = 0}\}$).
6. (5pt) If M, N are finitely generated, then $\operatorname{Supp} (M \otimes N) = \operatorname{Supp} M \cap \operatorname{Supp} N$.
7. (*This is optional.*) Give an example of an A-module such that $\operatorname{Supp} M$ is not a closed subset in $\operatorname{Spec} A$.

Exercises from class.

Problem 4.[5pt] Let $f \in A$. The canonical homomorphism $\phi : A \to A_f$ induces a continuous map $\phi^* : \operatorname{Spec} A_f \to \operatorname{Spec} A$. Show that ϕ^* induces a homeomorphism between $\operatorname{Spec} A_f$ and the principal open set X_f.

Problem 5.[5pt] Show that the canonical image of $\operatorname{Spec} A_{\mathfrak{p}}$ in $\operatorname{Spec} A$ is the intersection of all open subsets containing $\mathfrak{p}$.