

Homework 4 for 506, Spring 2009

due Monday, May 4

Throughout this homework, k will be a field.

Problem 1. Let $V \subset \mathbb{A}_k^n, W \subset \mathbb{A}_k^m$ be algebraic sets.

- (1) Show that $\phi : V \rightarrow W$ is a regular morphism if and only if there exists $F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ such that for any $v \in V$,

$$\phi(v) = (F_1(v), \dots, F_m(v)).$$

- (2) Prove that a regular morphism $\phi : V \rightarrow W$ is continuous in Zariski topology.

Problem 2. Let V, W be affine algebraic varieties. For $\phi : V \rightarrow W$ a regular morphism denote by $\phi^* : k[W] \rightarrow k[V]$ the map of algebras defined by the formula $\phi^*(f) = f \circ \phi$ for $f \in k[W]$.

- (1) Show that ϕ^* is an algebra homomorphism.
(2) Show that the correspondence $\phi \mapsto \phi^*$ defines a bijection between the set of regular morphisms between V and W and algebra homomorphisms between $k[W]$ and $k[V]$.
(3) Show that ϕ is a regular isomorphism if and only if ϕ^* is an algebra isomorphism.

Problem 3. Let $\mathfrak{F} : \underline{A\text{-mod}} \rightarrow \underline{B\text{-mod}}$ be a functor. Show that \mathfrak{F} is exact if and only if it takes (long) exact sequences to (long) exact sequences.

Problem 4. Prove one of the following two statements (choose either one to prove but be aware of both!)

- (1) Show that $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of A -modules if and only if for any A -module N ,

$$0 \longrightarrow \text{Hom}(M'', N) \longrightarrow \text{Hom}(M, N) \longrightarrow \text{Hom}(M', N)$$

is exact;

- (2) Show that $0 \longrightarrow N' \longrightarrow N \longrightarrow N''$ is an exact sequence of A -modules if and only if for any A -module M

$$0 \longrightarrow \text{Hom}(M, N') \longrightarrow \text{Hom}(M, N) \longrightarrow \text{Hom}(M, N'')$$

is exact.

In particular, you have shown that Hom is a left exact functor. Now, give an example showing Hom is not exact.

Problem 5. Let \mathfrak{a} be an ideal in the ring A contained in the Jacobson radical, M be an A -module, and N be a finitely generated A -module. Let $f : M \rightarrow N$ be a homomorphism of A -modules such that the induced homomorphism $\bar{f} : M/\mathfrak{a}M \rightarrow N/\mathfrak{a}N$ is surjective. Prove that f is surjective.

Problem 6. Calculate:

- (1) $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$
(2) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$

Problem 7. Prove the “Snake Lemma”: for any commutative diagram of A -modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \downarrow \phi' & & \downarrow \phi & & \downarrow \phi'' & & \\ 0 & \longrightarrow & N' & \xrightarrow{t} & N & \xrightarrow{s} & N'' & \longrightarrow & 0 \end{array}$$

the following sequence is exact:

$$0 \longrightarrow \text{Ker } \phi' \longrightarrow \text{Ker } \phi \longrightarrow \text{Ker } \phi'' \xrightarrow{\delta} \text{Coker } \phi' \longrightarrow \text{Coker } \phi \longrightarrow \text{Coker } \phi'' \longrightarrow 0.$$

Here,

$$\delta : \text{Ker } \phi'' \rightarrow \text{Coker } \phi'$$

is the *connecting homomorphism* defined as follows. Let $m'' \in \text{Ker } \phi''$. Then there exists $m \in M$ such that $m'' = g(m)$. Commutativity of the diagram together with the exactness of the bottom row imply that $\phi(m) = t(n')$ for some $n' \in N'$. We define $\delta(m'') = \overline{n'} \in N'/\text{Im } \phi' = \text{Coker } \phi'$. All the other maps in the sequence are naturally induced by the maps in the diagram.

Remark 1. Checking that δ is well-defined is **part** of the exercise!

Remark 2. This kind of argument is called “diagram chase” - tedious but very useful.