**Homework 3 for 506, Spring 2009**
due Friday, April 24

Throughout this homework, $k$ will be a field.

**Problem 1.** Let $X \subset \mathbb{A}^n$ be an algebraic set. Show that $X$ is irreducible if and only if $I(X)$ is prime.

**Problem 2.** Let $\mathfrak{a} = (XY, YZ, XZ) \subset k[X, Y, Z]$. Let $X = V(\mathfrak{a}) \subset \mathbb{A}^3$. Answer the following questions (you may assume $k = \bar{k}$ if it helps):

(1) Describe (or sketch) $X$.
(2) What is the dimension of $X$?
(3) How many irreducible components does $X$ have?
(4) Prove or disprove: $\mathfrak{a} = I(V(\mathfrak{a}))$.
(5) Show that $\mathfrak{a}$ cannot be generated by two elements.

Now let $\mathfrak{a}' = (XY, (X - Y)Z) \subset k[X, Y, Z]$, and let $X' = V(\mathfrak{a}')$. Describe $V(\mathfrak{a}')$ and calculate $\text{rad}(\mathfrak{a}')$.

**Problem 3.** Let $f, g \in k[X, Y]$ be irreducible polynomials, such that neither one is a multiple of the other. Show that $V((f, g))$ is a finite set.

**Problem 4.**
(1) Let $A \subset B \subset C$ be algebras such that $B$ is finite over $A$, and $C$ is finite over $B$. Show that $C$ is finite over $A$.
(2) Let $B$ be a finite $A$-algebra, and let $b \in B$. Show that $b$ is a root of a monic polynomial over $A$, that is, there exist $a_0, \ldots, a_{n-1} \in A$ such that

$$b^n + a_{n-1}b^{n-1} \cdots + a_1 b + a_0 = 0.$$

(3) Prove the converse: Let $b$ be a root of a monic polynomial over $A$, then $B = A[b]$ is a finite $A$-algebra.

**Problem 5.** Let $k$ be an infinite field, and let $f \in k[x_1, \ldots, x_n]$. Assume that $f \not\equiv \text{const}$. Show that $V(f) \neq \mathbb{A}^n$.

**Problem 6.** Let $R$ be an integral domain with a unique non-trivial prime ideal $\mathfrak{p}$, and let $K$ be the fraction field of $R$. Let $S = R/\mathfrak{p} \times K$. Define $\phi : R \to S$ as $\phi(x) = (\bar{x}, x)$ where $\bar{x}$ is the image of $x$ in the quotient $R/\mathfrak{p}$. Show that $\phi^* : \text{Spec}\, S \to \text{Spec}\, R$ is bijective but not a homeomorphism.