Solution of Problem 5, Homework 2

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Problem 5. Let R be a Noetherian ring.

1. Show that Spec R is a Noetherian space and describe the irreducible components of Spec R in terms of prime ideals of R.

2. Show that $\dim \operatorname{Spec} R = \operatorname{Krull} \dim R$.

3. Let $\mathfrak{p}_x \subset R$ be a prime ideal, and $x \in \operatorname{Spec} R$ be the corresponding point in Spec R. Express $\dim \overline{x} = \dim V(\mathfrak{p}_x)$ as an algebraic characteristic of the ideal $\mathfrak{p}_x$.

Lemma A: Suppose $I, J \subset R$ are ideals in R. Then
(a) $V(I) \subset V(J)$ if and only if $\operatorname{rad}(J) \subset \operatorname{rad}(I)$.
(b) $V(I) = V(J)$ if and only if $\operatorname{rad}(J) = \operatorname{rad}(I)$.

Proof:
(a) ($\Longrightarrow$) Assume $V(I) \subset V(J)$. By definition this means that if $P \subset R$ is a prime ideal such that $I \subset P$, then we also have that $J \subset P$. Therefore, if $x \in \operatorname{rad}(J)$, then x is in every prime ideal containing J, since the radical of J is in fact the intersection of all prime ideals containing J. Specifically, if P is a prime ideal such that $I \subset P$, then $J \subset P$, and so $x \in P$. Therefore, $x \in \operatorname{rad}(I)$ and $\operatorname{rad}(J) \subset \operatorname{rad}(I)$. $\square$

($\Longleftarrow$) This direction we already know. If $\operatorname{rad}(J) \subset \operatorname{rad}(I)$, then

$$V(I) = V(\operatorname{rad}(I)) \subset V(\operatorname{rad}(J)) = V(J).$$

$\blacksquare$

(b) ($\Longrightarrow$) $V(I) = V(J) \Rightarrow V(I) \subset V(J) \overset{\text{by L.A.}}{\Rightarrow} \operatorname{rad}(J) \subset \operatorname{rad}(I)$,
$V(I) = V(J) \Rightarrow V(J) \subset V(I) \overset{\text{by L.A.}}{\Rightarrow} \operatorname{rad}(I) \subset \operatorname{rad}(J)$.
Hence, $\operatorname{rad}(I) = \operatorname{rad}(J)$. $\square$

($\Longleftarrow$) This follows easily: if $\operatorname{rad}(J) = \operatorname{rad}(I)$, then

$$V(I) = V(\operatorname{rad}(I)) = V(\operatorname{rad}(J)) = V(J).$$

$\blacksquare$

Lemma B: Suppose $I \subset R$ is an ideal. Then $V(I)$ is irreducible if and only if $\mathrm{rad}(I)$ is prime.

Proof: ($\Longrightarrow$) Assume $V(I)$ is irreducible, and that $J, K \subset R$ are ideals such that $JK \subset \mathrm{rad}(I)$. Then $V(JK) \supset V(\mathrm{rad}(I)) = V(I)$. But $V(JK) = V(J) \cup V(K)$, so we have $V(I) \subset V(J) \cup V(K)$. The irreducibility of $V(I)$ implies that $V(I) \subset V(J)$ or $V(I) \subset V(J)$. By Lemma A, this implies that $\mathrm{rad}(J) \subset \mathrm{rad}(I)$ or $\mathrm{rad}(K) \subset \mathrm{rad}(I)$. Since $J \subset \mathrm{rad}(J)$ and $K \subset \mathrm{rad}(K)$, we thus have $J \subset \mathrm{rad}(I)$ or $K \subset \mathrm{rad}(I)$. Hence, $\mathrm{rad}(I)$ is prime. □

($\Longleftarrow$) Assume that $\mathrm{rad}(I)$ is prime and that $V(I) = V(J) \cup V(K) = V(JK)$ for ideals $J, K \subset R$. This implies by Lemma A that $\mathrm{rad}(I) \supset \mathrm{rad}(JK) = \mathrm{rad}(J) \cap \mathrm{rad}(K)$. Thus, since $\mathrm{rad}(I)$ is prime, we have that either $\mathrm{rad}(J) \subset \mathrm{rad}(I)$ or $\mathrm{rad}(K) \subset \mathrm{rad}(I)$. Of course, this implies (by Lemma A, if you want), that $V(I) \subset V(J)$ or $V(I) \subset V(K)$, showing that $V(I)$ is irreducible. ■

Solution of Problem 5. (1) To show that Spec R is Noetherian, suppose

$$\text{Spec } R = V(I_0) \supset V(I_1) \supset V(I_2) \supset \cdots$$

for ideals $I_j \subset R$. We need to show that this descending chain terminates.

Lemma A implies that we get a chain in R

$$\mathrm{rad}(I_0) \subset \mathrm{rad}(I_1) \subset \mathrm{rad}(I_2) \subset \cdots$$

Because R is Noetherian, this chain must terminate, say at

$$\mathrm{rad}(I_m) = \mathrm{rad}(I_{m+1}) = \cdots$$

Then

$$V(\mathrm{rad}(I_m)) = V(\mathrm{rad}(I_{m+1})) = \cdots$$
$$V(I_m) = V(I_{m+1}) = \cdots$$

so we have the original chain terminating. □

Now we will describe the irreducible components of Spec R in terms of prime ideals of R. We know by Problem 4 that if $\mathfrak{N}$ is the nilradical of R, then there are a finite number of minimal prime ideals $P_1, \dots, P_m$ over $\mathfrak{N}$. We will show that $V(P_i)$ are exactly the irreducible components of Spec R for $i = 1, \dots, m$. Before we do this, we will first show that every prime ideal $P \subset R$ contains some P_i.

For this, we'll use Zorn's lemma. Let P be a prime ideal, and let $\mathscr{P} = \{Q \text{ prime} : \mathfrak{N} \subset Q \subset P\}$. We know $P \in \mathscr{P}$, so $\mathscr{P}$ is nonempty. Then let $\mathscr{C}$ be a nonempty chain in $\mathscr{P}$. We know from last semester that the intersection of a chain of prime ideals is prime. Let $I = \cap_{Q \in \mathscr{C}} Q$ be this intersection, so I is prime. Also, since $\mathfrak{N} \subset Q$ for every $Q \in \mathscr{C}$, then $\mathfrak{N} \subset I$; hence, $I \in \mathscr{P}$. We have shown that every nonempty chain in $\mathscr{P}$ has a lower bound in $\mathscr{P}$, so Zorn's lemma implies that $\mathscr{P}$ has a minimal element, say J. To see that $J = P_j$ for some j, suppose $\mathfrak{N} \subsetneq J' \subset J$ such that J' is prime. Then $J' \in \mathscr{P}$ by definition. The minimality of J then implies that $J' = J$. Hence, J is minimal over $\mathfrak{N}$, so

$J = P_j$ for some j. Since $P_j = J \subset P$, we have that P contains some minimal prime ideal over $\mathfrak{N}$.

What we have shown is that every prime ideal in R contains some minimal prime ideal $P_1, \dots, P_m$ over $\mathfrak{N}$. Thus,

$$P_1 \cap \cdots \cap P_m \subset \bigcap_{P \subset R \text{ prime}} P = \mathfrak{N}.$$

Since each $P_i \supset \mathfrak{N}$, we also have $P_1 \cap \cdots \cap P_m \supset \mathfrak{N}$, so now

$$P_1 \cap \cdots \cap P_m = \mathfrak{N}.$$

We know from the proof of Problem 2 that Spec $R = V(\mathfrak{N})$. Thus,

$$\text{Spec } R = V(\mathfrak{N}) = V(P_1 \cap \cdots \cap P_m) = V(P_1) \cup \cdots \cup V(P_m).$$

Because each P_j is prime, it is therefore radical and $\text{rad}(P_j) = P_j$. So by Lemma B, we have that $V(P_j)$ is irreducible for each j. Thus, this gives us our irreducible decomposition of Spec R. Specifically, the irreducible components are precisely the varieties corresponding to the minimal prime ideals over the nilradical $\mathfrak{N}$. ■

(2) Suppose we have a chain of irreducible closed sets

$$V(I_0) \subsetneq V(I_1) \subsetneq \ldots \subsetneq V(I_n),$$

Then by Lemma A this translates to a chain of ideals

$$\text{rad}(I_0) \supsetneq \text{rad}(I_1) \supsetneq \ldots \supsetneq \text{rad}(I_n).$$

Note that we maintain the inequalities by part (b) of Lemma A. Since each $V(I_j)$ is irreducible, then each $\text{rad}(I_j)$ is prime by Lemma B. So this is a chain of prime ideals in R.

Conversely, suppose

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

is a chain of prime ideals in R. In Spec R, this translates to a chain

$$V(P_0) \supsetneq V(P_1) \supsetneq \cdots \supsetneq V(P_n).$$

Note again that we maintain the inequalities in this chain by part (b) of Lemma A. Also, each $V(P_j)$ is irreducible by Lemma B: P_j is prime, so $P_j = \text{rad}(P_j)$.

We have shown the following. For any chain of irreducible closed sets in Spec R, we can find a chain of the same length in R of prime ideals. Also, for any chain of prime ideals in R, we can find a chain of the same length in Spec R of irreducible closed sets. Therefore, $\dim \text{Spec } R = \text{Krull} \dim R$.

■

(3) We know that $\dim V(\mathfrak{p}_x)$ as a space is

$$\dim V(\mathfrak{p}_x) = \sup\{n : V(I_0) \subsetneq \cdots \subsetneq V(I_n) \subset V(\mathfrak{p}_x) \,|\, V(I_j) \text{ is irreducible}\}.$$

As in the proof of (2) above, any such chain corresponds to a chain in R (and vice-versa); so we have

$$\dim V(\mathfrak{p}_x) = \sup\{n : P_0 \supsetneq \cdots \supsetneq P_n \supset \mathfrak{p}_x, \text{ where each } P_i \text{ is prime}\} = \text{depth } \mathfrak{p}_x.$$

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