Homework 2 for 506, Spring 2009
due Friday, April 17

Problem 1. Describe $\operatorname{Spec} R$ for

(1) $R = \mathbb{Z}[x]$
(2) $R = \mathbb{R}[x]$

Problem 2. Show that $\operatorname{Spec} R$ is irreducible if and only if $\mathfrak{N}$ is prime.
Remark - Corollary. If R is an integral domain then $\operatorname{Spec} R$ is irreducible.

Problem 3. Let $X = \operatorname{Spec} R$, $f \in R$. Show that the principal open set X_f is quasi-compact. (Recall that quasi-compact means that any open cover has a finite subcover.)

Problem 4. Prove that if R is Noetherian, and $\mathfrak{a} \subset R$ is an ideal, then among the primes containing $\mathfrak{a}$ there are only finitely many that are minimal with respect to inclusion. These ideals are called *minimal prime ideals* over $\mathfrak{a}$.

Problem 5. Let R be a Noetherian ring.

(1) Show that $\operatorname{Spec} R$ is a Noetherian space and describe irreducible components of $\operatorname{Spec} R$ in terms of prime ideals of R.
(2) Show that $\dim \operatorname{Spec} R =$ Krull $\dim R$.
(3) Let $\mathfrak{p}_x \subset R$ be a prime ideal, and $x \in \operatorname{Spec} R$ be the corresponding point in $\operatorname{Spec} R$. Express $\dim \bar{x} = \dim V(\mathfrak{p}_x)$ as an algebraic characteristic of the ideal $\mathfrak{p}_x$.

Exercises from class.

Exercise about radicals.

Problem 6.

(1) $\mathfrak{a} \subset \operatorname{rad}(\mathfrak{a})$
(2) $\operatorname{rad}(\operatorname{rad}(\mathfrak{a})) = \operatorname{rad}(\mathfrak{a})$
(3) $\operatorname{rad}(\mathfrak{a}\mathfrak{b}) = \operatorname{rad}(\mathfrak{a}) \cap \operatorname{rad}(\mathfrak{b}) = \operatorname{rad}(\mathfrak{a} \cap \mathfrak{b})$
(4) $\operatorname{rad}(\mathfrak{a}) = (1)$ if and only if $\mathfrak{a} = (1)$
(5) $\operatorname{rad}(\mathfrak{a} + \mathfrak{b}) = \operatorname{rad}(\operatorname{rad}(\mathfrak{a}) + \operatorname{rad}(\mathfrak{b}))$
(6) $\mathfrak{p}$ is a prime ideal. Then $\operatorname{rad}(\mathfrak{p}^{n}) = \mathfrak{p}$ for any $n > 0$

Exercises about irreducible spaces.

Problem 7. Show that for a topological space X to be irreducible is equivalent to any of the following

(1) Any non-empty open subset is dense;
(2) for any two non-empty open subsets $V, W \subset X$, we have $V \cap W \neq \emptyset$.

Problem 9. Let X be a topological space.

(1) If $Y \subset X$ is irreducible, then $\overline{Y}$ is irreducible;
(2) any irreducible subset is contained in a maximal one.