Homework 1 for 506, Spring 2009
due Friday, April 10

R is a commutative ring with an identity element.

Problem 1. Let $R[x]$ be a polynomial ring with coefficients in R, and let $f = a_n x^n + \cdots + a_0 \in R[x]$. Prove the following:

[i] f is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent;
[ii] f is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent;
[iii] f is a zero divisor in $R[x]$ if and only if there exists $a \in R$ such that $af = 0$.

Problem 2. Show that in $R[x]$ the nilradical coincides with the Jacobson radical

Problem 3. Let $\mathfrak{N}$ be the nilradical of R. Show that the following are equivalent:

[i] R has only one prime ideal;
[ii] Any element of R is either nilpotent or a unit;
[iii] $R/\mathfrak{N}$ is a field.

Problem 4. Show that a local ring does not have non-trivial idempotents. (An idempotent is an element $a \in R$ such that $a^2 = a$. Any ring has two trivial idempotents: 0 and 1.)

Exercises from class.

Problem 5. Give an example of a ring R such that $R/\mathfrak{N}$ is not an integral domain.

Problem 6. Let $\mathfrak{a}, \mathfrak{b}$ be ideals in R. Show that if $\mathfrak{a} + \mathfrak{b} = (1)$ then $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$.

Exercises on operations with ideals.

Problem 7.

(1) $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$
(2) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subset \mathfrak{a}$
(3) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$
(4) $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) = \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$
(5) $(\mathfrak{a} : \sum_i \mathfrak{b}_i) = \bigcap_i (\mathfrak{a} : \mathfrak{b}_i)$