Solutions for the Final for 506, Spring 2009
Wednesday, June 10

Throughout, A is a commutative ring with identity.

Problem 1. Let S be a multiplicatively closed set in A, and M be a finitely generated A-module. Show that $S^{-1}(\mathrm{Ann}_A M) = \mathrm{Ann}_{S^{-1}A}(S^{-1}M)$.

Solution. Let $\{m_1, \ldots, m_n\}$ be generators of M. Then

$$\mathrm{Ann}_A(M) = \bigcap_1^n \mathrm{Ann}_A(m_i),$$

and similarly for $S^{-1}M$. Since localization commutes with finite intersections, it is therefore enough to prove the statement for a cyclic module.

Let $M = Am$ be a cyclic module generated by m. Then $M \simeq A/\mathfrak{a}$ where $\mathfrak{a} = \mathrm{Ann}_A(m)$. The converse also holds: if $M \simeq A/\mathfrak{a}$ for some ideal $\mathfrak{a}$, then M is a cyclic module (generated by 1 mod $\mathfrak{a}$) and $\mathfrak{a} = \mathrm{Ann}_A(M)$.

Now, if $M \simeq A/\mathfrak{a}$, then $S^{-1}M = S^{-1}A/S^{-1}\mathfrak{a}$ since localization commutes with quotients. Therefore, $S^{-1}M$ is also a cyclic module (in fact, generated by $m/1$) with $\mathrm{Ann}_{S^{-1}A}S^{-1}M = S^{-1}\mathfrak{a} = S^{-1}(\mathrm{Ann}_A M)$.

Problem 2.

(1) Let M, N be flat A-modules. Show that $M \otimes_A N$ is also flat.
(2) Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of A-modules, and assume that M'' is flat. Show that M is flat if and only if M' is flat.

Solution. (1). Let

$$0 \longrightarrow W' \longrightarrow W \longrightarrow W'' \longrightarrow 0$$

be any short exact sequence of A-modules. Since M is flat, the sequence

$$0 \longrightarrow W' \otimes M \longrightarrow W \otimes M \longrightarrow W'' \otimes M \longrightarrow 0$$

is exact. Since N is also flat, we further conclude that

$$0 \longrightarrow W' \otimes M \otimes N \longrightarrow W \otimes M \otimes N \longrightarrow W'' \otimes M \otimes N \longrightarrow 0$$

is exact. Hence, tensoring with $M \otimes N$ is exact. By definition, $M \otimes N$ is flat.

(2). Tensoring with any module N, we get a long exact sequence:

$$\mathrm{Tor}_2(M'', N) \longrightarrow \mathrm{Tor}_1(M', N) \longrightarrow \mathrm{Tor}_1(M, N) \longrightarrow \mathrm{Tor}_1(M'', N) \longrightarrow$$

$$M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \longrightarrow 0\,.$$

If M'' is flat, then $\mathrm{Tor}_2(M'', N) = \mathrm{Tor}_1(M'', N) = 0$. Hence, the map $\mathrm{Tor}_1(M', N) \to \mathrm{Tor}_1(M, N)$ in the long exact sequence above is an isomorphism (since it is surrounded by two zeros). Therefore, $\mathrm{Tor}_1(M', N) = 0$ if and only if $\mathrm{Tor}_1(M, N) = 0$. Since vanishing of Tor_1 provides a criterion for flatness by one of the homework problems, we conclude that M' is flat if and only if M is flat.

Problem 3. Let p be a prime. Describe the following topological spaces (points, irreducible components and dimension):

(1) $\mathrm{Spec}\,\mathbb{Z}_{(p)}$,
(2) $\mathrm{Spec}\,\mathbb{Z}_{(p)}[x]$.

Solution. (1). $\operatorname{Spec}\mathbb{Z}_{(p)}$ has two points: $\{(0), p\mathbb{Z}_{(p)}\}$; three closed sets: $\emptyset$, $p\mathbb{Z}_{(p)}$ and the whole thing; one closed point: $p\mathbb{Z}_{(p)}$; one irreducible component; and dimension 1. This can be shown directly using properties of localization. Alternatively, we can use the fact that $\mathbb{Z}_{(p)}$ is a DVR, corresponding to the discrete valuation ν_p on $\mathbb{Q}$. Then the description above is implies by the structure theorem for DVRs.

(2). Let $S = \mathbb{Z} - (p)$. Note that S is still a multiplicatively closed set in $\mathbb{Z}[x]$. Moreover, $\mathbb{Z}_{(p)}[x] = S^{-1}\mathbb{Z}[x]$. Hence, we can use the result from a homework problem which described $\operatorname{Spec}\mathbb{Z}[x]$.

Recall that $\mathbb{Z}[x]$ has 3 types of non-zero prime ideals:

(1) (q), where q is a prime number,
(2) $(f(x))$, where f is irreducible polynomial of degree at least 1,
(3) $(q, f(x))$, where q is prime, f is monic irreducible and $f(x) \bmod q$ is still irreducible and has the same degree as f.

When we localize with respect to $S = A - (p)$, we have one-to-one correspondence between prime ideals in the localization and prime ideals in $\mathbb{Z}[x]$ which do not intersect S. Let's analyze what this correspondence does to the list above:

(1) $(q) \cap S \neq \emptyset$ unless $q = p$. Hence, all these ideals die except for (p).
(2) This type does not overlap with S; so they all survive and give different prime ideals in $\mathbb{Z}_{(p)}[x]$.
(3) Finally, the only ideals of this type that "survive" are the ones for which $q = p$.

Therefore, the following is a complete list of prime ideals in $\mathbb{Z}_{(p)}[x]$:

(1) (0)
(2) (p)
(3) $(f(x))$, where f is an irreducible polynomial of degree at least 1
(4) $(p, f(x))$, where f is monic irreducible and $f(x) \bmod p$ is still irreducible and has the same degree as f.

The longest chains of proper prime ideals are of the form $(0) \subset (p) \subset (p, f(x))$ and $(0) \subset (f(x)) \subset (p, f(x))$ (here, $(p, f(x))$ is still proper unless $f(x) \bmod p$ is a constant). Therefore, $\dim \operatorname{Spec}\mathbb{Z}_{(p)}[x] = 2$ (this also follows from the Krull dimension theorem). The space is irreducible since $\mathbb{Z}_{(p)}[x]$ is an integral domain.

Note that $(f(x)$ is maximal if and only if $(p, f(x)) = \mathbb{Z}_{(p)}[x]$ which happens if and only if $f(x) = pg(x) + a$, $a \in \mathbb{Z}$, $(p, a) = 1$. Hence, the following is a complete list of closed points:

(1) $(f(x))$, where $f(x)$ is an irreducible polynomial of degree at least 1 and $f(x) \bmod p$ is a non-zero constant
(2) $(p, f(x))$, where f is monic irreducible and $f(x) \bmod p$ is still irreducible and has the same degree as f.

and the following is a complete list of irreducible closed sets:

$\boxed{\dim = 2}$ $V((0))$,

$\boxed{\dim = 1}$ $V((p))$,
$V((f(x))$ where $f(x)$ is irreducible and $f(x) \bmod p$ is a polynomial of degree at least 1 in $\mathbb{F}_p[x]$

$\boxed{\dim = 0}$ $V((p, f(x)))$, where f is monic irreducible of degree at least 1 and $f(x)$ mod p is still irreducible and has the same degree as f,
$V((f(x))$, where $f(x) \bmod p$ is a non-zero constant

Note. The problem did NOT ask you to figure out which ideals were maximal and to write down explicitly all closed sets. So if you did not, you should not lose any points.

Problem 4. Let A be a Noetherian ring, $\mathfrak{p}$ be a prime ideal in A, and $S_{\mathfrak{p}} = A - \mathfrak{p}$. Let

$$\mathfrak{p}^{(n)} = (\mathfrak{p}^n : S_{\mathfrak{p}}) = \{a \in A \,|\, \exists\, s \in S_{\mathfrak{p}} \text{ such that } as \in \mathfrak{p}^n\}$$

be the n^{th} symbolic power of $\mathfrak{p}$. Show that $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary component of the primary decomposition of $\mathfrak{p}^n$.

Solution. Since A is Noetherian, there exists a minimal primary decomposition $\mathfrak{p}^n = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_m$. Localize at $\mathfrak{p}$. Since localization commutes with primary decomposition, we get

$$S_{\mathfrak{p}}^{-1}\mathfrak{p}^n = (S_{\mathfrak{p}}^{-1}\mathfrak{p})^n = S_{\mathfrak{p}}^{-1}\mathfrak{q}_1 \cap \ldots \cap S_{\mathfrak{p}}^{-1}\mathfrak{q}_m.$$

Since $S_{\mathfrak{p}}^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$ is the maximal ideal in the local ring $A_{\mathfrak{p}}$, the ideal $(S_{\mathfrak{p}}^{-1}\mathfrak{p})^n$ is $S_{\mathfrak{p}}^{-1}\mathfrak{p}$-primary. By the uniqueness theorem for the primary decomposition, we must have $S^{-1}\mathfrak{q}_i = (S_{\mathfrak{p}}^{-1}\mathfrak{p})^n$ for some i, and $S_{\mathfrak{p}}^{-1}\mathfrak{q}_j = A_{\mathfrak{p}}$ for all $j \neq i$. Therefore, for $i \neq j$, the component $\mathfrak{q}_j$ is not contained in $\mathfrak{p}$ and therefore is not $\mathfrak{p}$-primary. The component $\mathfrak{q}_i$ corresponds to $(S_{\mathfrak{p}}^{-1}\mathfrak{p})^n$ via the one-to-one correspondence between the primary ideals in A contained in $\mathfrak{p}$ and primary ideals in the localization $A_{\mathfrak{p}}$. Hence, $\mathfrak{q}_i$ is the restriction of $(S_{\mathfrak{p}}^{-1}\mathfrak{p})^n$ via the canonical map $\phi : A \to A_{\mathfrak{p}}$. Hence,

$$\mathfrak{q}_i = \phi^{-1}(S_{\mathfrak{p}}^{-1}\mathfrak{p}^n) = \{a \in A \,|\, \frac{a}{1} \in S_{\mathfrak{p}}^{-1}\mathfrak{p}^n\} = \{a \in A \,|\, \frac{a}{1} \sim \frac{b}{s} \text{ for some } b \in \mathfrak{p}^n, s \in S\} =$$
$$\{a \in A \,|\, t(as - b) = 0 \text{ for some } b \in \mathfrak{p}^n, s, t \in S\} = \{a \in A \,|\, as = b \text{ for some } b \in \mathfrak{p}^n, s \in S\}$$
$$= \{a \in A \,|\, as \in \mathfrak{p}^n \text{ for some } s \in S\} = (\mathfrak{p}^n : S) = \mathfrak{p}^{(n)}.$$

We used that if $t(as - b) = 0$ then $t(as - b) \in \mathfrak{p}$ and, therefore, $as - b \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $t \in S_{\mathfrak{p}} = A - \mathfrak{p}$. We also have $\mathrm{rad}(\mathrm{q_i}) = \mathfrak{p}$ (use that localization commutes with radicals or refer to a homework problem where we proved that $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary). This finishes the proof.