WORKSHEET ON ARTINIAN (AND NOETHERIAN) MODULES, MATH 505, 2019

DUE WEDNESDAY, MARCH 13

Throughout this worksheet A is a ring with 1, not necessarily commutative.

Definition 1.1. Let A be a ring, and M be a (left) A-module. We say that M satisfies a *descending chain condition* if any decreasing chain of submodules

$$M_1 \supset M_2 \supset \ldots \supset M_{i-1} \supset M_i \supset \ldots$$

stabilizes.

Lemma 1.2. *Prove that the following conditions on an A-module M are equivalent:*

1. *M satisfies the descending chain condition*
2. *Any non-empty set of submodules of M has a minimal element*

Proof. Problem 1 [10pts] □

Definition 1.3.

1. Let A be a ring, and M be a (left) A-module. We say that M is Artinian if it satisfies the *descending chain condition*.
2. A ring A is *Artinian* if it is an Artinian module over itself.

Provide justifications for the following examples.

Example 1.4 (10pts)**.**

1. $\mathbb{Q}$ is neither Noetherian nor Artinian as a $\mathbb{Z}$-module.
2. Let p be fixed prime number. Let $G < \mathbb{Q}/Z$ be a subgroup consisting of all elements of order p^n for all non negative integers n. Then G is an Artinian $\mathbb{Z}$-module but not a Noetherian one (i.e., satisfies the descending chain condition but not the ascending chain condition).

Proposition 1.5. *Let* $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ *be a short exact sequence of A-modules. Then M_2 is Artinian if and only if M_1, M_3 are Artinian.*

Proof. Problem 3 [10pts]. □

Proposition 1.6. *Let A be an Artinian ring, and M be a finitely generated A-module. Then M is Artinian.*

Proof. Problem 4 [10pts]. □

Problem 5 [20pts]. Give examples of the following:

1. A Noetherian ring which has a non-Noetherian subring.
2. A ring which is neither Noetherian, nor Artinian.
3. A Noetherian ring which is not Artinian.
4. A ring which is both Noetherian and Artinian.