WORKSHEET ON FROBENIUS NORMAL FORM

DUE WEDNESDAY, FEBRUARY 10, 2016

Throughout this worksheet k will be field. Make a note of one significant difference with the Jordan canonical form: k is NOT assumed to be algebraically closed.

Definition 1.1. Let V be a finite dimensional vector space over k, and let $\mathcal{L} : V \to V$ be a k-linear operator. Consider a ring homomorphism

$$\phi : k[t] \to \mathrm{End}_k(V)$$

defined by sending t to $\mathcal{L}$ and extending k-linearly. Let $\mathrm{Ker}\,\phi$ be the kernel ideal, and let $q_{\mathcal{L}}(t)$ be a monic polynomial in $k[t]$ which generates $\mathrm{Ker}\,\phi$. Then $q_{\mathcal{L}}(t)$ is the **minimal polynomial** of $\mathcal{L}$.

Note that $q_{\mathcal{L}}(t)$ exists since $k[t]$ is a PID and is unique since we assume it is monic.

Lemma 1.2. *Let $A = k[t]$, and let M be a cyclic torsion A-module (hence, M is finite dimensional as a k vector space). Let $q(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0$ be the minimal polynomial of t considered as a linear operator on M. Prove that M has a basis with respect to which the matrix representing the action of t has the following form:*

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \cdots & \cdots & \cdots & \ldots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix} \tag{1}$$

Terminology. This matrix is sometimes called *the companion* matrix of the polynomial $q(t)$.

Proof. Exercise. □

Lemma 1.3. *Show that the characteristic polynomial of the matrix in Lemma 1.2 is $q(t)$.*

Theorem 1.4. *Let $A = k[t]$ and let M be a finitely generated torsion A-module. Prove that there exist non-constant monic polynomials $q_1(t), \ldots, q_m(t)$, determined uniquely, such that $q_1(t) \,|\, q_2(t) \,|\, \ldots \,|\, q_m(t)$, and*

$$M \simeq A/(q_1(t)) \oplus A/(q_2(t)) \oplus \cdots \oplus A/(q_m(t)).$$

Proof. Exercise. □

Definition 1.5. The polynomials $q_1, \ldots, q_m$ are called the **invariant factors** of M.

Theorem 1.6. *Let V be a finite dimensional k-vector space, and let $\mathcal{L} : V \to V$ be a k-linear transformation of V. Let $\chi_{\mathcal{L}}(t)$ be the characteristic polynomial of $\mathcal{L}$.*

(1) *There exist uniquely determined non-constant monic polynomials $q_1(t), \ldots, q_m(t)$ such that*
(2) (a) *$q_i(t) \,|\, q_{i+1}(t)$ for $1 \le i \le m-1$.*
  (b) *$q_m(t)$ is the minimal polynomial of $\mathcal{L}$.*
  (c) *$\chi_{\mathcal{L}}(t) = \epsilon q_1(t)q_2(t)\ldots q_m(t)$ for $\epsilon \in k$.*

(3) *There exists a basis of V such that the matrix of $\mathcal{L}$ with respect to this basis is a block matrix with m blocks $A_1, \ldots, A_m$, and each block A_i is in the companion matrix for q_i.*

Proof. Exercise. □