Homework 1 for 505, Winter 2016

due Wednesday, January 13

Problem 1: Inseparable extensions. Recall that an extension $K/k$ is separable if and only if every element $\alpha \in K$ is separable over $k$ if and only if $\text{Irr}(\alpha, k)$ does not have multiple roots for any $\alpha \in K$. Also recall that any extension of a field of zero characteristic is separable.

We say that an algebraic extension $K/k$ is purely inseparable if it does not contain any separable elements aside from $k$. Let $k$ be a field of positive characteristic $p$.

The questions below can be done in any order.

(1) Let $K/k$ be a purely inseparable extension. Show that for any $\alpha \in K$ there exists $n \in \mathbb{N}$ such that $\alpha^{p^n} \in k$.

(2) Let $K/k$ be a finite purely inseparable extension. Show that $[K : k] = p^n$.

(3) Let $K/k$ be a finite algebraic extension. Show that there exists a subextension $k \subset E \subset K$ such that

(a) $E/k$ is separable,

(b) $K/E$ is purely inseparable,

(c) $[E : k] = [K : k]_s$

(d) $[K : k] = p^n[K : k]_s$

Here, $[K : k]_s$ is the separable degree of the extension $K/k$. Note: $[K : E] = p^n = \frac{[K : k]}{[K : k]_s}$ is called the inseparable degree of $K/k$.

Problem 2. Compute Galois groups of the splitting fields of the following polynomials

(1) $X^3 - X - 1$ over $\mathbb{Q}$

(2) $X^3 - 10$ over $\mathbb{Q}(\sqrt{2})$

(3) $X^3 - X - t$ over $\mathbb{C}(t)$

Problem 3. Let $f(X) = X^4 - 4X^2 - 1 \in \mathbb{Q}[X]$. Determine the splitting field and its Galois group (over $\mathbb{Q}$) and describe the lattice of subfields and the corresponding lattice of subgroups for the Galois group.

Problem 4. (Symmetric group revisited.) Recall that $\{g_1, \ldots, g_n\}$ is called a “minimal system” of generators of a group $G$ if the elements $\{g_1, \ldots, g_n\}$ generate $G$ but no proper subset of them does.

Also recall that $S_n$ always has a minimal system of generators which consists of a transposition and a cycle. In fact, $S_n$ is generated by $(1 \ldots n)$ and $(12)$.

(1) Show that if $n = p$ is prime, then ANY cycle of length $p$ and ANY transposition form a minimal system of generators.

(2) Give a counterexample to the previous statement for a non-prime $n$.

(3) Show that $S_n$ has a minimal system of $k$ generators for any $k$, $2 \leq k \leq n - 1$.

(4) (BONUS - optional) Is it true for $k = n$?
Problem 5. [Last problem from the final]. Let $F(x_1, x_2, \ldots, x_n)$ be the field of rational functions on $n \geq 2$ variables over a field $F$. Consider the elements

$$X_1 := x_1 x_2, \quad X_2 := x_2 x_3, \ldots, \quad X_n := x_n x_1.$$ 

(1) Suppose $n$ is even. Show that $X_1, X_2, \ldots, X_n$ are algebraically dependent. [Find a short relation of degree $n/2$. The case $n = 4$ should reveal the idea.]

(2) Suppose $n$ is odd. Show that $X_1, X_2, \ldots, X_n$ are algebraically independent.