Throughout this worksheet $A$ is a ring with 1, not necessarily commutative.

**Definition 1.1.** Let $A$ be a ring, and $M$ be a (left) $A$-module. We say that $M$ satisfies a *descending chain condition* if any decreasing chain of submodules

$$M_1 \supset M_2 \supset \ldots \supset M_{i-1} \supset M_i \supset \ldots$$

stabilizes.

**Lemma 1.2.** Prove that the following conditions on an $A$-module $M$ are equivalent:

1. $M$ satisfies the descending chain condition
2. Any non-empty set of submodules of $M$ has a minimal element

**Proof.** Exercise. □

**Definition 1.3.**

1. Let $A$ be a ring, and $M$ be a (left) $A$-module. We say that $M$ is Artinian if it satisfies the descending chain condition.
2. A ring $A$ is Artinian if it is an Artinian module over itself.

Provide justifications for the following examples.

**Example 1.4.**

1. $\mathbb{Z}$ is Noetherian but not Artinian (as a ring).
2. Let $p$ be fixed prime number. Let $G < \mathbb{Q}/\mathbb{Z}$ be a subgroup consisting of all elements of order $p^n$ for some $n \geq 0$. Then $G$ is an Artinian $\mathbb{Z}$-module but not a Noetherian one (i.e., satisfies the descending chain condition but not the ascending chain condition).

**Proposition 1.5.** Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be a short exact sequence of $A$-modules. Then $M_2$ is Artinian if and only if $M_1$, $M_3$ are Artinian.

**Proof.** Exercise. □

**Proposition 1.6.** Let $A$ be an Artinian ring, and $M$ be a finitely generated $A$-module. Then $M$ is Artinian.

**Proof.** Exercise. □