Homework 7 for 505, Winter 2010
due Wednesday, February 24

Tags: Composition series, Noetherian spaces
Throughout, A is a ring with identity, all modules are left modules.

Problem 1. Show that an A-module M has a composition series if and only if it is Noetherian and Artinian.

Def. Let $\phi : A-\text{mod} \rightarrow M$ be a function from A-modules to an abelian monoid $M$. It is called additive if for any short exact sequence

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

of A-modules we have $\phi(M_2) = \phi(M_1) + \phi(M_3)$.

Def. Let $M$ be an A-module. If $M$ has a composition series, then we denote by $\ell(M)$ the length of the series. Otherwise, $\ell(M) = +\infty$.

Problem 2. Show that the length function $\ell : A-\text{mod} \rightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is additive.

Def. A topological space $X$ is Noetherian if any increasing chain of open subspaces of $X$ stabilizes (equivalently, if any set of open subsets has a maximal element). Again, equivalently, if any decreasing chain of closed subsets stabilizes.

Example. Consider the real line with Zariski topology: closed sets are finite subsets of points, plus the whole line and the empty set. Convince yourself this is a Noetherian space.

Problem 3. Show that a subspace of a Noetherian space is Noetherian.

Def. Recall that a space is called quasi-compact if any open covering has a finite subcovering.

Problem 4. Show that the following are equivalent for a topological space $X$:

1. $X$ is Noetherian
2. Any open subspace of $X$ is quasi-compact
3. Any subspace of $X$ is quasi-compact

Problem 5. Let $k$ be a field, and $A = M_n(k)$. Show that the left ideal $L_j \subset A$, which consists of all matrices with zeros everywhere off the $j$th column is a simple A-module.