Homework 6 for 505, Winter 2010
due Wednesday, February 10

Tags: linear algebra, modules over PID, Noetherian and Artinian modules

PART 1: Modules over PID.

Problem 1. Let $V$ be a two-dimensional real vector space, and let $\mathcal{L}: V \to V$ be a linear operator described by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

(1) Show that $V$ is a cyclic $\mathbb{R}[x]$-module (where $x$ acts via $\mathcal{L}$, as usual)
(2) Find the rational canonical form of the matrix $A$.

Problem 2. Let $A$ be a P.I.D., let $a$ be a non-zero element of $A$, and let $M = A/(a)$. Let $p$ be a prime in $A$, and let $p^n$ be the maximal power of $p$ dividing $a$. Prove that

$$p^{\ell-1}M/p^\ell M \cong \begin{cases} A/(p) & \text{if } \ell \leq n \\ 0 & \text{if } \ell > n \end{cases}$$

Problem 3.

(1) Let $A$ be a $3 \times 3$ matrix with rational coefficients satisfying $A^6 = I$. List all similarity classes of $A$.
(2) Same thing for a matrix $A$ over $\mathbb{F}_2$.

Problem 4. Find the number of conjugacy classes and a representative for each conjugacy class for the finite (simple!) group $GL_3(\mathbb{F}_2)$.

PART 2: Noetherian modules.

Problem 5. Give examples of the following:

(1) A Noetherian ring which has a non-Noetherian subring.
(2) A ring which is neither Noetherian, nor Artinian.
(3) A Noetherian ring which is not Artinian.
(4) A ring which is both Noetherian and Artinian.

At this point you should be asking yourself a question ...

Problem 6. Let $A$ be a commutative ring, and $M$ be a Noetherian $A$-module. Let $\text{ann}(M) = \{a \in A | am = 0 \text{ for any } m \in M\}$, the annihilator of $M$. Prove that $A/\text{ann}(M)$ is a Noetherian ring.