Solutions/sketches for the Final for 505, Winter 2010

Wednesday, March 17, in-class

Problem 1. Let $A$ be a principal ideal domain, and let $F$ be a finitely generated free module. Show that any submodule of $F$ is free.

Solution. Consult your notes.

Problem 2. Prove a special case of the uniqueness part of the Artin-Wedderburn theorem. Let $D, D'$ be division rings. Show that $M_n(D) \simeq M_m(D')$ (as rings) if and only if $n = m, D \simeq D'$.

Solution. Let $A = M_n(D)$. Let $L_i$ be a left ideal of $M_n(D)$ consisting of matrices with zeros everywhere except in the $i$th column. By the fact stated in class (and proved in the special case when $D$ is a field), $L_i$, for $1 \leq i \leq n$, is a simple module for $M_n(D)$. Moreover, $L_1 \simeq L_i$ for any $i, 1 \leq i \leq n$, as $M_n(D)$-modules (the isomorphism just takes the 1st column to the $i$th column), and we have $A = M_n(D) = L_1 \oplus L_2 \oplus \ldots \oplus L_n \simeq L_1^\oplus n$.

Similarly, $A = M_n(D') = L'_1 \oplus \ldots \oplus L'_m \simeq (L'_1)^\oplus m$, where $L'_i$ is the left ideal consisting of matrices in $M_m(D')$ with zeros everywhere except in the $i$th column.

Note that a direct sum decomposition $A \simeq L_1^\oplus n$ yields a composition series for $A$ as an $A$-module:

$$0 \subset L \subset L \oplus L \subset \ldots \subset L^\oplus n$$

with $n$ composition factors, each one isomorphic to $L$. Since we have constructed two direct sum decompositions: $A \simeq L_1^\oplus n$ and $A \simeq (L'_1)^\oplus m$, the Jordan-Holder theorem implies that $n = m$ and $L_1 \simeq L'_1$ as $A$-modules. Therefore, $D \simeq \text{End}_A(L_1, L_1) \simeq \text{End}_A(L'_1, L'_1) \simeq D'$.

Problem 3. Let $F$ be a finite field. Prove that for any positive integer $n$ there exists an irreducible polynomial $f(x) \in F[x]$ of degree $n$.

Solution. Let $F = \mathbb{F}_{p^n}$, and let $L = \mathbb{F}_{p^{mn}}$. By the theory for finite fields, $[L : F] = n$. Also, $L/F$ is a separable extension since finite fields are perfect. By the primitive element theorem, there exists $\alpha \in L$ such that $L = F(\alpha)$. Let $f(x) = \text{Irr}(\alpha, F)$. Then $\deg f(x) = [L : F] = n$, and $f$ is irreducible by construction.

Problem 4. Let $G$ be a finite abelian group. Show that any irreducible representation of $G$ over an algebraically closed field $k$ is one–dimensional.

Solution 1 (Using Artin–Wedderburn theorem). Let $J(kG)$ be the Jacobson radical of the group algebra $kG$. First, we observe that $kG/J(kG)$ is a semi–simple algebra. Indeed, since $kG$ is a finite–dimensional $k$-algebra, it is Artinian. Therefore, $kG/J(kG)$ is also Artinian, since a quotient of an Artinian ring is Artinian. Moreover, $J(kG/J(kG)) = 0$ (a classical property of the Jacobson radical proved in homework). Another result we proved in class says that if $A$ is an Artinian ring such that $J(A) = 0$ then $A$ is semi–simple. Hence, $kG/J(kG)$ is semi–simple.

By Artin-Wedderburn, $kG/J(kG)$ is a product of matrix rings of the form $M_n(D)$ where $D$ is a finite–dimensional division algebra over $k$. By the “Tiny Wedderburn theorem”\footnote{Tiny Wedderburn theorem: the only finite dimensional division algebra over an algebraically closed field $k$ is $k$ itself}, there are no non-trivial finite-dimensional division algebras over algebraically closed fields. Hence, $M_n(D) = M_n(k)$. We conclude that $kG/J(kG)$ is a product of matrix rings over $k$. Now commutativity of $kG$ implies that all these
matrix rings are trivial. We conclude \( kG/J(kG) \simeq k \oplus k \oplus \ldots \oplus k \). Since any simple \( kG/J(kG) \) module must appear in this decomposition, we get that all simple \( kG/J(kG) \)-modules are one-dimensional.

Now let \( L \) be an irreducible representation of \( G \). Then \( L \) is a simple \( kG \)-module. Let \( m \subset kG \) be the annihilator of \( L \). Then \( J(kG) \subset m \) by the definition of the Jacobson radical. Hence, \( J(kG) \) acts trivially on \( L \). We conclude that the action of \( kG \) on \( L \) factors through the action of \( kG/J(kG) \). In other words, \( L \) is a simple module for \( kG/J(kG) \). By the discussion above, \( L \) is one-dimensional.

**Solution 2** (Using Schur’s lemma). Let \( L \) be an irreducible representation of \( G \). Let \( s \in G \), and define \( \rho_s : L \to L \) via \( \rho_s(m) = sm \) for any \( m \in L \). This is evidently a \( k \)-linear operator. We claim that \( \rho_s \) is a \( G \)-invariant map (that is, \( \rho_s \in \text{End}_{kG}(L) \)). Indeed, let \( g \in G, m \in L \). Then
\[
\rho_s(gm) = gsm = g \rho_s(m)
\]
(This is where we need commutativity of \( G \)) Since \( k \) is algebraically closed, Schur’s lemma together with the “Tiny Wedderburn theorem” implies that \( \text{End}_{kG}(L) = k \).

Therefore, \( \rho_s \in \text{End}_{kG}(L) = k \) is simply a multiplication by a scalar.

Let \( m \) be any element in \( L \). Then \( sm = \rho_s m \in km \subset L \). Hence, any element in \( L \) generates a \( G \)-invariant subspace. Since \( L \) is irreducible, we must have \( L = km \). Therefore, \( L \) is one-dimensional.

**Solution 3** (Jim Stark.) Any homomorphism \( \chi : G \to k^* \) is a one dimensional representation of \( G \) whose character is exactly \( \chi \). As \( G \) is abelian it has \( |G| \) conjugacy classes and hence \( |G| \) irreducible representations; therefore, to prove the proposition it suffices to exhibit \( |G| \) different homomorphisms from \( G \) to \( k^* \).

By the fundamental theorem of abelian groups we take \( G \) to be a direct product of cyclic groups
\[
G = \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_n.
\]
For each \( 1 \leq i \leq n \) let \( \omega_i \in k \) be a primitive \((d_i)\)th root of unity; this we can do because \( k \) is algebraically closed. Given any \( t = (t_1, \ldots, t_n) \) in \( G \) define \( \chi_t : G \to k^* \) by
\[
\chi_t(x_1, \ldots, x_n) = \omega_1^{t_1 x_1} \cdots \omega_n^{t_n x_n}.
\]
Observe that \( t_i x_i \) is in \( \mathbb{Z}/d_i \) but \( \omega_i^{t_i} = 1 \) so this is a well defined map and, by elementary ring identities, a homomorphism. Assume \( s, t \in G \) with \( s \neq t \). Then \( s \) and \( t \) differ in some coordinate, say the \( i \)th. Let \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) be the element of \( G \) that has zeros everywhere save a one in the \( i \)th coordinate. Then
\[
\chi_s(e_i) = \omega_i^{t_i} \quad \text{and} \quad \chi_t(e_i) = \omega_i^{s_i}.
\]
As \( s_i \neq t_i \) and \( \omega_i \) is primitive we have \( \omega_i^{t_i} \neq \omega_i^{s_i} \); hence, \( \chi_s \neq \chi_t \). This shows that \( \chi_t \), as \( t \) ranges over \( G \), gives \( |G| \) distinct homomorphisms from \( G \) to \( k^* \) and completes the proof of the proposition. \( \square \)

**Problem 5.**

1. Let \( V \) be a complex representation of the symmetric group \( S_n \). Let \( m = \text{dim} V \), let \( \chi \) be the character of \( V \), and let \( s \in S_n \) be a transposition. What are the possible values of \( \chi(s) \)?
2. Answer the same question in the following special case: \( V \) is an irreducible representation of \( S_3 \).

**Solution.** a). Let \( \rho : S_n \to \text{GL}_m \) be the representation of \( S_n \) on \( V \). Since \( s^2 = 1 \), \( \rho(s)^2 = \text{Id} \). Hence, for any eigenvalue \( \lambda \) of \( \rho(s) \), we must have \( \lambda^2 = 1 \). Therefore,
$\lambda = \pm 1$. This implies that the possible values for $\text{Tr}(\rho(s))$ are

$$-m, -m + 2, \ldots, m - 2, m,$$

$m + 1$ total values. We note that all of them can be realized. Indeed, let $V = \text{triv}^\oplus \ell \oplus \text{sgn}^\oplus (m - \ell)$. Then $\chi_V(s) = \ell \chi_{\text{triv}}(s) + (m - \ell) \chi_{\text{sgn}}(s) = \ell - (m - \ell) = 2\ell - m$.

When $\ell$ ranges from 0 to $m$, the value of $\chi_V(s)$ ranges through the list above.

b). There are three irreducible representations of $S_3$: triv, sgn and the 2-dimensional standard representation $W$. If we write down the character table, we can just read off the values for $s = (12)$:

<table>
<thead>
<tr>
<th></th>
<th>triv</th>
<th>sgn</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>id</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(12)</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(123)</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

In particular, in dimension 2 only one value, 0, is realized by an irreducible representation. For 2 and $-2$ we have to take triv$^\oplus 2$ and sgn$^\oplus 2$ respectively.