Homework 6 for 504, Fall 2018
due Wednesday, November 21

Problem 1. Let G be a p-group such that $G/Z(G)$ is cyclic. Show that G is abelian

Problem 2. Let G is a p -group, and H be a proper subgroup. Show that H is a proper subgroup in $N_G(H)$.

Problem 3. Let $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\mathrm{Id}, -\mathrm{Id}\}$ be the projective special linear group over $\mathbb{Z}$.

- Show that $\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$.
- Show that $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to an amalgamated product of two cyclic groups: $\mathrm{SL}_2(\mathbb{Z}) \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

Hint. Try $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ for the generators of your cyclic groups.

Problem 4. Denote by E_{ij} an $n \times n$ matrix where the entry (i, j) is 1 and all the other entries are 0. Let I_n be the identity matrix. Let B_n be the group of upper triangular matrices over a field F where $F = \mathbb{R}$ or $F = \mathbb{F}_p$, U_n be the subgroup of matrices with 1's on tehe diagonal, and $U_n^{(k)}$ be the subgroup of U_n of all matrices with the first k super diagonals being zero.

(1) Show that $I_n + aE_{ij}$, $1 \le i < j \le n$, generate U_n. Hint: Try downward induction of $U_n^{(k)}$.
(2) Show that $I_n + aE_{ii}$, $1 \le i \le n$, generate the group of diagonal matrices T_n.
(3) Let $a \neq 0$. Compute $(I_n + aE_{ii})(I_n + bE_{j\ell})(I + aE_{ii})^{-1}$ (The answer should depend on i, j, ℓ.)
(4) Show that $[B_n, U_n] = U_n$.
(5) Conclude that B_n is not a nilpotent group.

Problem 5.

(1) Show that in the category of abelian groups, the free object on a set I is $\bigoplus\limits_I \mathbb{Z}$
(2) Show that the following hom sets are bijective:
    (a) $\mathrm{Hom}_{Ab}(\prod\limits_{\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \cong \bigoplus\limits_{\mathbb{N}} \mathbb{Z}$
    (b) $\mathrm{Hom}_{Ab}(\bigoplus\limits_{\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \cong \prod\limits_{\mathbb{N}} \mathbb{Z}$