Homework 1 for 504, Fall 2018
due Wednesday, October 3

A general note about homework: I am happy to discuss homework before it is due, either in person or by e-mail. Your comments/feedback on difficulty level, length, manageablity, interest, and anything else (homework related) you can think of are welcome. Working in groups is fine but solutions must be written individually.

Problem 1. Show that an identity element and an inverse in a group are unique.

Problem 2. Let G be a set with two binary operations, denoted $*$ and $\circ$, and a fixed element e such that

(1) e is the identity for both operations
(2) $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$

Show that these two operations coincide and moreover that they are associative and commutative.

Problem 3. *Cyclic groups.*
(1). Let $a \in G$ be an element of order n. Show that if $a^m = e$ then $n|m$.
(2). Prove that a subgroup of a cyclic group is cyclic.
(3). Let $f : G \to G'$ be a group homomorphism, and assume that G is cyclic. Show that $\operatorname{Im} f$ is a cyclic subgroup of G'.
(4). Prove that all subgroups of $\mathbb{Z}$ have the form $m\mathbb{Z}$ for $m \in \mathbb{Z}$.
(5). Classify all subgroups of $\mathbb{Z}/m\mathbb{Z}$.
(6). Show that all finitely generated subgroups of $\mathbb{Q}$ are cyclic and isomorphic to $\mathbb{Z}$.

Problem 4. *Quaternions.*
Let $V = \mathbb{R}^3$ and set $C = \mathbb{R} \times V$. (As a real vector space, C may be identified with $\mathbb{R}^4$.) Define a product on C by

$$(a, u)(b, v) = (ab - u \cdot v; av + bu + u \times v).$$

(C is called the quaternion algebra). Define the *norm* $N(a; u) = a^2 + |u|^2$.

(1) Show that the product defined above is associative but not commutative.
(2) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$ for $\alpha, \beta \in C$.
(3) Show that the set of nonzero elements of C is a group under multiplication. (Hence, (2) shows that $N : C^* \to \mathbb{R}^*$ is a homomorphism.)
(4) Show that elements $\alpha \in C$ with integer coefficients and such that $N(\alpha) = 1$ form a subgroup of order 8 in C^*. This is the *quaternion group.*

Problem 5. Give examples:

(1) of a group G with two minimal sets of generators of different cardinality,
(2) of an infinite group generated by two elements.