1. Elementary symmetric polynomials

Definition 1.1. Let $R$ be a ring (commutative, with unit). A polynomial $f \in R[x_1, \ldots, x_n]$ is symmetric if for any $\sigma \in S_n$, $f(\sigma(x_1), \ldots, \sigma(x_n)) = f(x_1, \ldots, x_n)$

Alternatively, define the action of $S_n$ on $R[x_1, \ldots, x_n]$ via

$$\sigma \circ f(x_1, \ldots, x_n) = f(\sigma(x_1), \ldots, \sigma(x_n)).$$

The symmetric polynomials are invariants of this action - the polynomials for which the stabilizer is the entire group $S_n$.

Example 1.2. Let $n = 3$. Then $x_1^{17} + x_2^{17} + x_3^{17}$, $x_1x_2^{16} + x_2x_3^{16} + x_3x_1^{16}$ are symmetric whereas $x_1^2x_2^2x_3^3$ is not.

Consider the polynomial $P(t) = (t - x_1)(t - x_2) \cdots (t - x_n)$ in $R[x_1, \ldots, x_n][t]$. Let

$$P(t) = t^n - s_1(x_1, \ldots, x_n)t^{n-1} + s_2(x_1, \ldots, x_n)t^{n-2} - \ldots + (-1)^n s_n(x_1, \ldots, x_n)$$

Definition 1.3. Polynomials $s_i(x_1, \ldots, x_n)$, $1 \leq i \leq n$, are called the elementary symmetric polynomials.

Observe that $P(t)$ is clearly invariant under the action of $S_n$. Hence, the elementary symmetric polynomials are, in fact, symmetric. Of course, one can write them down explicitly:

$s_1 = x_1 + \ldots + x_n$

$s_2 = x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n$

$\ldots$

$s_n = x_1 \ldots x_n$

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and denote by $x^\alpha$ the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. We’ll say that $x^\alpha > x^\beta$ if $\alpha > \beta$ in lexicographical order. If $f$ is a polynomial in $R[x_1, \ldots, x_n]$ then the multidegree of $f$ is the degree $\alpha$ of the maximal monomial in $f$. The degree of a monomial $x^\alpha$ is $\alpha_1 + \ldots + \alpha_n$. The degree of a polynomial $f$ is the degree of its highest monomial.

Observe that any symmetric polynomial containing $x^\alpha$ must contain $\sum_{\sigma \in S_n} x_1^{\sigma(\alpha_1)} \ldots x_n^{\sigma(\alpha_n)}$.

Definition 1.4. A polynomial $f$ is called homogeneous if $f$ is a sum of monomials of the same degree.

Note that elementary symmetric polynomials are homogeneous and determined by a multidegree $\alpha$ which consists of only 0’s and 1’s.

Theorem 1.5. (= Problem 1) Let $f(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ be a symmetric polynomial. Then there exists a polynomial $F \in R[x_1, \ldots, x_n]$ such that $f(x_1, \ldots, x_n) = F(s_1, \ldots, s_n)$.

In other words, any symmetric polynomial can be expressed in terms of elementary ones.
Example 1.6. $x_1^3 + x_2^3 + x_3^3 = s_1^3 - 3s_1s_2 + 3s_3$.

Definition 1.7. We say that $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ are algebraically independent if there does not exist $F \in R[x_1, \ldots, x_m]$ such that $F(f_1, \ldots, f_m) = 0$.

Theorem 1.8. (=Problem 2). Prove that elementary symmetric polynomials on $n$ variables are algebraically independent.

The combination of these two results is sometimes referred to as the “Fundamental Theorem of symmetric polynomials”:

Theorem 1.9. The ring of invariants of the polynomial ring on $n$ variables under the action of the symmetric group is a polynomial ring on the elementary symmetric polynomials:

$$R[x_1, \ldots, x_n]^{S_n} \simeq R[s_1, \ldots, s_n].$$

$R[x_1, \ldots, x_n]^{S_n}$ is called the ring of symmetric polynomials.

2. Newton identities

Let $p_k(x_1, \ldots, x_n) = x_1^k + \ldots + x_n^k$. By the previous theorem, $p_k$ can be expressed in terms of elementary symmetric polynomials. Explicit formulas can be obtained recursively from the Newton Identities:

$$ks_k = \sum_{i=1}^{k} (-1)^{i-1} s_{k-i}p_i$$

Problem 3.

(1) Prove Newton identities. Assume $s_0 = 1$.

(2) Assume $R$ is a field of characteristic 0. Show that $\{p_1, \ldots, p_n\}$ are algebraically independent generators of the ring of symmetric polynomials $R[x_1, \ldots, x_n]^{S_n}$.

Corollary 2.1. Let $t_1, \ldots, t_n$ be all roots (counted with multiplicity and, possibly, complex) of a polynomial of degree $n$ with real coefficients. Then $t_1^k + \ldots + t_n^k$ is a real number for any $k$.

Problem 4. Let $A$ be a real-valued matrix. Show that the characteristic polynomial $\chi(A)$ can be expressed exclusively in terms of $\text{Tr}(A^k)$ for $k \geq 1$. 