Postulates of Neutral Geometry

Postulate 1 (The Set Postulate). *Every line is a set of points, and there is a set of all points called **the plane**.*

Postulate 2 (The Existence Postulate). *There exist at least three distinct noncollinear points.*

Postulate 3 (The Unique Line Postulate). *Given any two distinct points, there is a unique line that contains both of them.*

Postulate 4 (The Distance Postulate). *For every pair of points A and B, the distance from A to B is a nonnegative real number determined by A and B.*

Postulate 5 (The Ruler Postulate). *For every line ℓ, there is a bijective function $f\colon \ell \to \mathbb{R}$ with the property that for any two points $A, B \in \ell$, we have*

$$AB = |f(B) - f(A)|.$$

Postulate 6 (The Plane Separation Postulate). *For any line ℓ, the set of all points not on ℓ is the union of two disjoint subsets called the **sides of** ℓ. If A and B are distinct points not on ℓ, then A and B are on the same side of ℓ if and only if $\overline{AB} \cap \ell = \varnothing$.*

Postulate 7 (The Angle Measure Postulate). *For every angle $\angle ab$, the measure of $\angle ab$ is a real number in the closed interval $[0, 180]$ determined by $\angle ab$.*

Postulate 8 (The Protractor Postulate). *For every ray $\vec{r}$ and every point P not on $\overleftrightarrow{r}$, there is a bijective function $g\colon \mathrm{HR}(\vec{r}, P) \to [0, 180]$ that assigns the number 0 to $\vec{r}$ and the number 180 to the ray opposite $\vec{r}$, and such that if $\vec{a}$ and $\vec{b}$ are any two rays in $\mathrm{HR}(\vec{r}, P)$, then*

$$m\angle ab = \left|g(\vec{b}) - g(\vec{a})\right|.$$

Postulate 9 (The SAS Postulate). *If there is a correspondence between the vertices of two triangles such that two sides and the included angle of one triangle are congruent to the corresponding sides and angle of the other triangle, then the triangles are congruent under that correspondence.*

Theorems of Neutral Geometry

Theorem 3.1. *Every line contains infinitely many distinct points.*

Corollary 3.2 (Incidence Axiom 4). *Given any line, there are at least two distinct points that lie on it.*

Lemma 3.3 (Ruler Sliding Lemma). *Suppose ℓ is a line and $f\colon \ell \to \mathbb{R}$ is a coordinate function for ℓ. Given a real number c, define a new function $f_1\colon \ell \to \mathbb{R}$ by $f_1(X) = f(X) + c$ for all $X \in \ell$. Then f_1 is also a coordinate function for ℓ.*

Lemma 3.4 (Ruler Flipping Lemma). *Suppose ℓ is a line and $f\colon \ell \to \mathbb{R}$ is a coordinate function for ℓ. If we define $f_2\colon \ell \to \mathbb{R}$ by $f_2(X) = -f(X)$ for all $X \in \ell$, then f_2 is also a coordinate function for ℓ.*

Theorem 3.5 (Ruler Placement Theorem). *Suppose ℓ is a line and A, B are two distinct points on ℓ. Then there exists a coordinate function $f\colon \ell \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) > 0$.*

Theorem 3.6 (Properties of Distances). *If A and B are any two points, their distance has the following properties:*

(a) $AB = BA$.

(b) $AB = 0$ *if and only if* $A = B$.

(c) $AB > 0$ *if and only if* $A \neq B$.

Theorem 3.7 (Symmetry of Betweenness of Points). *If A, B, C are any three points, then $A * B * C$ if and only if $C * B * A$.*

Theorem 3.8 (Betweenness Theorem for Points). *Suppose A, B, and C are points. If $A * B * C$, then $AB + BC = AC$.*

Theorem 3.9 (Hilbert's Betweenness Axiom). *Given three distinct collinear points, exactly one of them lies between the other two.*

Corollary 3.10 (Consistency of Betweenness of Points). *Suppose A, B, C are three points on a line ℓ. Then $A * B * C$ if and only if $f(A) * f(B) * f(C)$ for every coordinate function $f\colon \ell \to \mathbb{R}$.*

Theorem 3.11 (Partial Converse to the Betweenness Theorem for Points). *If A, B, and C are three distinct collinear points such that $AB+BC=AC$, then $A*B*C$.*

Theorem 3.12. *Suppose A and B are distinct points. Then*

$$\overleftrightarrow{AB} = \{P : P*A*B \text{ or } P=A \text{ or } A*P*B \text{ or } P=B \text{ or } A*B*P\}.$$

Lemma 3.13. *If $A_1, A_2, \ldots, A_k$ are distinct collinear points, then $A_1*A_2*\cdots*A_k$ if and only if $A_k*\ldots A_2*A_1$.*

Theorem 3.14. *Given any k distinct collinear points, they can be labeled $A_1, \ldots, A_k$ in some order such that $A_1*A_2*\cdots*A_k$.*

Theorem 3.15. *Suppose A,B,C are points such that $A*B*C$. If P is any point on $\overleftrightarrow{AB}$, then one and only one of the following relations holds:*

$$\begin{gathered} P=A, \quad P=B, \quad P=C, \\ P*A*B*C, \quad A*P*B*C, \quad A*B*P*C, \quad A*B*C*P. \end{gathered} \tag{3.1}$$

Theorem 3.16. *Suppose A,B,C,D are four distinct points. If any of the following pairs of conditions holds, then $A*B*C*D$:*

$$\begin{aligned} &A*B*C \text{ and } B*C*D; \qquad \text{or} \\ &A*B*C \text{ and } A*C*D; \qquad \text{or} \\ &A*B*D \text{ and } B*C*D. \end{aligned}$$

On the other hand, if $A*B*C*D$, then all of the following conditions are true:

$$A*B*C, \qquad A*B*D, \qquad A*C*D, \qquad \text{and} \qquad B*C*D.$$

Lemma 3.17. *If A and B are two distinct points, then $\overline{AB} \subseteq \overleftrightarrow{AB}$.*

Theorem 3.18 (Segment Extension Theorem). *If $\overline{AB}$ is any segment, there exist points $C,D \in \overleftrightarrow{AB}$ such that $C*A*B$ and $A*B*D$.*

Lemma 3.19. *Suppose S and T are sets of points in the plane with $S \subseteq T$. Every passing point of S is also a passing point of T.*

Theorem 3.20. *Suppose A and B are distinct points. Then A and B are extreme points of $\overline{AB}$, and every other point of $\overline{AB}$ is a passing point.*

Corollary 3.21 (Consistency of Endpoints of Segments). *Suppose A and B are distinct points, and C and D are distinct points, such that $\overline{AB} = \overline{CD}$. Then either $A=C$ and $B=D$, or $A=D$ and $B=C$.*

Theorem 3.22 (Euclid's Common Notions for Segments).

(*a*) **Transitive Property of Congruence:** *Two segments that are both congruent to a third segment are congruent to each other.*

(*b*) **Segment Addition Theorem:** *Suppose A,B,C,A',B',C' are points such that $A*B*C$ and $A'*B'*C'$. If $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.*

(*c*) **Segment Subtraction Theorem:** *Suppose A,B,C,A',B',C' are points such that $A*B*C$ and $A'*B'*C'$. If $\overline{AC} \cong \overline{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$, then $\overline{BC} \cong \overline{B'C'}$.*

(*d*) **Reflexive Property of Congruence:** *Every segment is congruent to itself.*

(*e*) **The Whole Segment is Greater Than the Part:** *If A, B, and C are points such that $A*B*C$, then $AC > AB$.*

Lemma 3.23 (Coordinate Representation of a Segment). *Suppose A and B are distinct points, and $f : \overleftrightarrow{AB} \to \mathbb{R}$ is a coordinate function for $\overleftrightarrow{AB}$. Then*

$$\begin{aligned} \overline{AB} &= \{P \in \overleftrightarrow{AB} : f(A) \le f(P) \le f(B)\} \qquad \text{if } f(A) < f(B); \\ \overline{AB} &= \{P \in \overleftrightarrow{AB} : f(A) \ge f(P) \ge f(B)\} \qquad \text{if } f(A) > f(B). \end{aligned}$$

Theorem 3.24. *If A,B,C are points such that $A*B*C$, then the following set equalities hold:*

(*a*) $\overline{AB} \cup \overline{BC} = \overline{AC}$.

(b) $\overline{AB} \cap \overline{BC} = \{B\}$.

Corollary 3.25. *If $A * B * C$, then $\overline{AB} \subseteq \overline{AC}$ and $\overline{BC} \subseteq \overline{AC}$.*

Lemma 3.26. *Let $\overline{AB}$ be a segment, and let M be a point. The following statements are all equivalent to each other:*

(a) *M is a midpoint of $\overline{AB}$ (i.e., $M \in \operatorname{Int}\overline{AB}$ and $MA = MB$).*

(b) *$M \in \overleftrightarrow{AB}$ and $MA = MB$.*

(c) *$M \in \overline{AB}$ and $AM = \frac{1}{2}AB$.*

Theorem 3.27 (Existence and Uniqueness of Midpoints). *Every segment has a unique midpoint.*

Theorem 3.28. *Every segment contains infinitely many distinct points.*

Theorem 3.29 (Euclid's Postulate 3). *Given two distinct points O and A, there exists a circle whose center is O and whose radius is OA.*

Lemma 3.30. *Suppose A and B are distinct points, and P is a point on the line $\overleftrightarrow{AB}$. Then $P \notin \overrightarrow{AB}$ if and only if $P * A * B$.*

Lemma 3.31. *Suppose A and B are distinct points. Then $\overline{AB} \subseteq \overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.*

Lemma 3.32 (Coordinate Representation of a Ray). *Suppose A and B are distinct points, and $f\colon \overleftrightarrow{AB} \to \mathbb{R}$ is a coordinate function for $\overleftrightarrow{AB}$. Then*

$$\begin{aligned} \overrightarrow{AB} &= \{P \in \overleftrightarrow{AB} : f(P) \geq f(A)\} \qquad \text{if } f(A) < f(B); \\ \overrightarrow{AB} &= \{P \in \overleftrightarrow{AB} : f(P) \leq f(A)\} \qquad \text{if } f(A) > f(B). \end{aligned}$$

Lemma 3.33 (Representation of a Ray in Adapted Coordinates). *Suppose A and B are distinct points, and $f\colon \overleftrightarrow{AB} \to \mathbb{R}$ is a coordinate function adapted to $\overrightarrow{AB}$. If P is any point on $\overleftrightarrow{AB}$, then $P \in \overrightarrow{AB}$ if and only if $f(P) \geq 0$, and $P \in \operatorname{Int}\overrightarrow{AB}$ if and only if $f(P) > 0$.*

Lemma 3.34 (Ordering Lemma for Points). *Suppose $\vec{a}$ is a ray starting at a point A, and B and C are interior points of $\vec{a}$ such that $AC > AB$. Then $A * B * C$.*

Theorem 3.35 (Segment Construction Theorem). *Suppose $\vec{a}$ is a ray starting at A, and r is a positive real number. Then there exists a unique point C in the interior of $\vec{a}$ such that $AC = r$.*

Corollary 3.36 (Unique Point Theorem). *Suppose $\vec{a}$ is a ray starting at A, and C and C' are points in $\operatorname{Int}\vec{a}$ such that $AC = AC'$. Then $C = C'$.*

Corollary 3.37 (Euclid's Segment Cutoff Theorem). *If $\overline{AB}$ and $\overline{CD}$ are segments with $CD > AB$, there is a unique point E in the interior of $\overline{CD}$ such that $\overline{CE} \cong \overline{AB}$.*

Theorem 3.38 (Rays with the Same Endpoint). *Suppose $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are rays with the same endpoint.*

(a) *If A, B, and C are collinear, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear.*

(b) *If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear, then they are either equal or opposite, but not both.*

(c) *If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, then $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$ and $\overrightarrow{AB} \cup \overrightarrow{AC} = \overleftrightarrow{AC}$.*

(d) *$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are equal if and only if they have an interior point in common.*

(e) *$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays if and only if $C * A * B$.*

Theorem 3.39 (Opposite Ray Theorem). *Every ray has a unique opposite ray.*

Theorem 3.40. *Let $\overrightarrow{AB}$ be the ray from A through B. Then A is the only extreme point of $\overrightarrow{AB}$.*

Corollary 3.41 (Consistency of Endpoints of Rays). *If A, B are distinct points and C, D are distinct points such that $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.*

Theorem 3.42. *Suppose A and B are two distinct points. Then the following set equalities hold:*

(a) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.

(b) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

Theorem 3.43 (Properties of Sides of Lines). *Suppose ℓ is a line.*

(a) *Both sides of ℓ are nonempty sets.*

(b) *If A and B are distinct points not on ℓ, then A and B are on opposite sides of ℓ if and only if $\overline{AB} \cap \ell \neq \varnothing$.*

Lemma 3.44 (The Y-Lemma). *Suppose ℓ is a line, A is a point on ℓ, and B is a point not on ℓ. Then every interior point of $\overrightarrow{AB}$ is on the same side of ℓ as B, and $\overrightarrow{AB} \subseteq \mathrm{CHP}(\ell, B)$.*

Lemma 3.45 (The X-Lemma). *Suppose $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are opposite rays, and ℓ is a line that intersects $\overleftrightarrow{AB}$ only at O. Then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ lie on opposite sides of ℓ.*

Theorem 3.46. *Suppose ℓ is a line, A is a point on ℓ, and B is a point not on ℓ. Then $\overrightarrow{AB} = \overleftrightarrow{AB} \cap \mathrm{CHP}(\ell, B)$.*

Theorem 3.47. *If $S_1, \ldots, S_k$ are convex subsets of the plane, then $S_1 \cap \cdots \cap S_k$ is convex.*

Theorem 3.48. *Every line is a convex set.*

Theorem 3.49. *Every segment is a convex set.*

Theorem 3.50. *Every open or closed half-plane is a convex set.*

Theorem 3.51. *Every ray is a convex set.*

Theorem 4.1. *If $\angle ab$ is a proper angle, then the common endpoint of $\vec{a}$ and $\vec{b}$ is the only extreme point of $\angle ab$.*

Corollary 4.2 (Consistency of Vertices of Proper Angles). *If $\angle AOB$ and $\angle A'O'B'$ are equal proper angles, then $O = O'$.*

Theorem 4.3 (Properties of Angle Measures). *Suppose $\angle ab$ is any angle.*

(a) $m\angle ab = m\angle ba$.
(b) $m\angle ab = 0°$ *if and only if $\angle ab$ is a zero angle.*
(c) $m\angle ab = 180°$ *if and only if $\angle ab$ is a straight angle.*
(d) $0° < m\angle ab < 180°$ *if and only if $\angle ab$ is a proper angle.*

Theorem 4.4 (Euclid's Postulate 4). *All right angles are congruent.*

Theorem 4.5 (Angle Construction Theorem). *Let O be a point, let $\vec{a}$ be a ray starting at O, and let x be a real number such that $0 < x < 180$. On each side of $\overleftrightarrow{a}$, there is a unique ray $\vec{b}$ starting at O such that $m\angle ab = x$.*

Corollary 4.6 (Unique Ray Theorem). *Let $\vec{a}$ be a ray starting at a point O. If $\vec{b}$ and $\vec{b}'$ are rays starting at O and lying on the same side of $\overleftrightarrow{a}$ such that $m\angle ab = m\angle ab'$, then $\vec{b}$ and $\vec{b}'$ are the same ray.*

Theorem 4.7 (Symmetry of Betweenness of Rays). *If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are rays with a common endpoint, then $\vec{a} * \vec{b} * \vec{c}$ if and only if $\vec{c} * \vec{b} * \vec{a}$.*

Theorem 4.8 (Betweenness Theorem for Rays). *If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are rays such that $\vec{a} * \vec{b} * \vec{c}$, then $m\angle ab + m\angle bc = m\angle ac$.*

Theorem 4.9. *If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are rays with a common endpoint, no two of which are collinear, and all lying in a single half-rotation, then exactly one of them lies between the other two.*

Corollary 4.10 (Consistency of Betweenness of Rays). *Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be distinct rays with a common endpoint, no two of which are collinear, and all lying in a single half-rotation. If $\mathrm{HR}(\vec{r}, P)$ is any half-rotation containing all three rays and g is a corresponding coordinate function, then $\vec{a} * \vec{b} * \vec{c}$ if and only if $g(\vec{a}) * g(\vec{b}) * g(\vec{c})$.*

Theorem 4.11 (Euclid's Common Notions for Angles).

(a) **Transitive Property of Congruence:** *Two angles that are both congruent to a third angle are congruent to each other.*

(b) **Angle Addition Theorem:** *Suppose $\vec{a}, \vec{b}, \vec{c}, \vec{a}', \vec{b}', \vec{c}'$ are rays such that $\vec{a} * \vec{b} * \vec{c}$ and $\vec{a}' * \vec{b}' * \vec{c}'$. If $\angle ab \cong \angle a'b'$ and $\angle bc \cong \angle b'c'$, then $\angle ac \cong \angle a'c'$.*

(c) **Angle Subtraction Theorem:** *Suppose $\vec{a},\vec{b},\vec{c},\vec{a}',\vec{b}',\vec{c}'$ are rays such that $\vec{a}*\vec{b}*\vec{c}$ and $\vec{a}'*\vec{b}'*\vec{c}'$. If $\angle ac \cong \angle a'c'$ and $\angle ab \cong \angle a'b'$, then $\angle bc \cong \angle b'c'$.*

(d) **Reflexive Property of Congruence:** *Every angle is congruent to itself.*

(e) **The Whole Angle is Greater Than the Part:** *If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are rays such that $\vec{a}*\vec{b}*\vec{c}$, then $m\angle ac > m\angle ab$.*

Theorem 4.12. *If $\vec{a},\vec{b},\vec{c}$ are rays such that $\vec{a}*\vec{b}*\vec{c}$, then $\angle ab$ and $\angle bc$ are adjacent angles.*

Theorem 4.13. *Supplements of congruent angles are congruent, and complements of congruent angles are congruent.*

Theorem 4.14 (Linear Pair Theorem). *If two angles form a linear pair, then they are supplementary.*

Corollary 4.15. *If two angles in a linear pair are congruent, then they are both right angles.*

Theorem 4.16 (Partial Converse to the Linear Pair Theorem). *If two adjacent angles are supplementary, then they form a linear pair.*

Theorem 4.17 (Vertical Angles Theorem). *Vertical angles are congruent.*

Theorem 4.18 (Partial Converse to the Vertical Angles Theorem). *Suppose $\vec{a}$ and $\vec{c}$ are opposite rays starting at O, and $\vec{b}$ and $\vec{d}$ are rays starting at O and lying on opposite sides of $\overleftrightarrow{a}$. If $\angle ab \cong \angle cd$, then $\vec{b}$ and $\vec{d}$ are opposite rays.*

Theorem 4.19 (Linear Triple Theorem). *If $\angle ab$, $\angle bc$, and $\angle cd$ form a linear triple, then their measures add up to* $180°$.

Theorem 4.20. *The interior of a proper angle is a convex set.*

Lemma 4.21. *Suppose $\angle AOC$ is a proper angle, and $\overrightarrow{OB}$ is a ray that lies in the interior of $\angle AOC$. Then $\overrightarrow{OA}*\overrightarrow{OB}*\overrightarrow{OC}$.*

Theorem 4.22 (The 360 **Theorem).** *Suppose $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three distinct rays with the same endpoint, such that no two of the rays are collinear and none of the rays lies in the interior of the angle formed by the other two. Then*

$$m\angle ab + m\angle bc + m\angle ac = 360°.$$

Lemma 4.23 (Interior Lemma). *Suppose $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three rays with the same endpoint, no two of which are collinear. Then $\vec{b}$ lies in the interior of $\angle ac$ if and only if $\vec{a}*\vec{b}*\vec{c}$.*

Corollary 4.24 (Restatement of the 360 Theorem in Terms of Betweenness). *Suppose $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three distinct rays with the same endpoint, such that no two of the rays are collinear and none of the rays lies between the other two. Then*

$$m\angle ab + m\angle bc + m\angle ac = 360°.$$

Lemma 4.25 (Ordering Lemma for Rays). *Suppose $\vec{a}$, $\vec{b}$, and $\vec{c}$ are rays with the same endpoint, such that $\vec{b}$ and $\vec{c}$ are on the same side of $\overleftrightarrow{a}$ and $m\angle ab < m\angle ac$. Then $\vec{a}*\vec{b}*\vec{c}$.*

Lemma 4.26 (Adjacency Lemma). *Suppose $\angle ab$ and $\angle bc$ are adjacent angles sharing the common side $\vec{b}$. If either of the following conditions holds, then $\vec{a}*\vec{b}*\vec{c}$.*

(a) *$\vec{a}$, $\vec{b}$, and $\vec{c}$ all lie in a single half-rotation.*

(b) $m\angle ab + m\angle bc < 180°$.

Theorem 4.27 (Betweenness vs. Betweenness). *Suppose ℓ is a line, O is a point not on ℓ, and A,B,C are three distinct points on ℓ. Then $A*B*C$ if and only if $\overrightarrow{OA}*\overrightarrow{OB}*\overrightarrow{OC}$.*

Theorem 4.28 (Existence and Uniqueness of Angle Bisectors). *Every proper angle has a unique angle bisector.*

Theorem 4.29 (Four Right Angles Theorem). *If ℓ and m are perpendicular lines, then ℓ and m form four right angles.*

Theorem 4.30 (Constructing a Perpendicular). *Let ℓ be a line and let P be a point on ℓ. Then there exists a unique line m that is perpendicular to ℓ at P.*

Theorem 5.1 (Consistency of Triangle Vertices). *If $\triangle ABC$ is a triangle, the only extreme points of $\triangle ABC$ are A, B, and C. Thus if $\triangle ABC = \triangle A'B'C'$, then the sets $\{A,B,C\}$ and $\{A',B',C'\}$ are equal.*

Theorem 5.2 (Pasch's Theorem). *Suppose $\triangle ABC$ is a triangle and ℓ is a line that does not contain any of the points A, B, or C. If ℓ intersects one of the sides of $\triangle ABC$, then it also intersects another side.*

Corollary 5.3. *If $\triangle ABC$ is a triangle and ℓ is a line that does not contain any of the points A, B, or C, then either ℓ intersects exactly two sides of $\triangle ABC$ or it intersects none of them.*

Theorem 5.4 (The Crossbar Theorem). *Suppose $\triangle ABC$ is a triangle and $\overrightarrow{AD}$ is a ray between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then the interior of $\overrightarrow{AD}$ intersects the interior of $\overline{BC}$.*

Theorem 5.5 (Transitive Property of Congruence of Triangles). *Two triangles that are both congruent to a third triangle are congruent to each other.*

Theorem 5.6 (ASA Congruence). *If there is a correspondence between the vertices of two triangles such that two angles and the included side of one triangle are congruent to the corresponding angles and side of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 5.7 (Isosceles Triangle Theorem). *If two sides of a triangle are congruent to each other, then the angles opposite those sides are congruent.*

Theorem 5.8 (Converse to the Isosceles Triangle Theorem). *If two angles of a triangle are congruent to each other, then the sides opposite those angles are congruent.*

Corollary 5.9. *A triangle is equilateral if and only if it is equiangular (that is, all three of its angles are congruent).*

Theorem 5.10 (Triangle Copying Theorem). *Suppose $\triangle ABC$ is a triangle, and $\overline{DE}$ is a segment congruent to $\overline{AB}$. On each side of $\overleftrightarrow{DE}$, there is a point F such that $\triangle DEF \cong \triangle ABC$.*

Theorem 5.11 (Unique Triangle Theorem). *Suppose $\overline{DE}$ is a segment, and F and F' are points on the same side of $\overleftrightarrow{DE}$ such that $\triangle DEF \cong \triangle DEF'$. Then $F = F'$.*

Theorem 5.12 (SSS Congruence). *If there is a correspondence between the vertices of two triangles such that all three sides of one triangle are congruent to the corresponding sides of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 5.13 (Exterior Angle Inequality). *The measure of an exterior angle of a triangle is strictly greater than the measure of either remote interior angle.*

Corollary 5.14. *The sum of the measures of any two angles of a triangle is less than* $180°$.

Corollary 5.15. *Every triangle has at least two acute angles.*

Theorem 5.16 (Scalene Inequality). *Let $\triangle ABC$ be a triangle. Then $AC > BC$ if and only if $m\angle B > m\angle A$.*

Corollary 5.17. *In any right triangle, the hypotenuse is strictly longer than either leg.*

Theorem 5.18 (Triangle Inequality). *If A, B, and C are noncollinear points, then $AC < AB + BC$.*

Theorem 5.19 (General Triangle Inequality).

(*a*) *If A, B, C are any three points (not necessarily distinct), then $AC \leq AB + BC$, and equality holds if and only if $A = B$, $B = C$, or $A * B * C$.*

(*b*) *If $n \geq 3$ and $A_1, \dots, A_n$ are any n points (not necessarily distinct), then $A_1A_n \leq A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n$.*

Corollary 5.20 (Converse to the Betweenness Theorem for Points). *If A, B, and C are three distinct points and $AB + BC = AC$, then $A * B * C$.*

Theorem 5.21 (Hinge Theorem). *Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$. Then $m\angle A > m\angle D$ if and only if $BC > EF$.*

Theorem 5.22 (AAS Congruence). *If there is a correspondence between the vertices of two triangles such that two angles and a nonincluded side of one triangle are congruent to the corresponding angles and side of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 5.24 (ASS Alternative Theorem). *Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ (the hypotheses of ASS). Then $\angle C$ and $\angle F$ are either congruent or supplementary.*

Theorem 5.25 (Angle-Side-Longer-Side Congruence). *Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ (the hypotheses of ASS), and assume in addition that $BC \geq AB$. Then $\triangle ABC \cong \triangle DEF$.*

Theorem 5.26 (HL Congruence). *If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and one leg of another, then the triangles are congruent under that correspondence.*

Chapter 7 theorems

Theorem 7.1 (Dropping a Perpendicular). *Suppose ℓ is a line and A is a point not on ℓ. Then there exists a unique line that contains A and is perpendicular to ℓ.*

Theorem 7.2. *Suppose $\overrightarrow{AB}$ is a ray, P is a point not on $\overleftrightarrow{AB}$, and F is the foot of the perpendicular from P to $\overleftrightarrow{AB}$.*

(*a*) *$F = A$ if and only if $\angle PAB$ is a right angle.*

(*b*) *$F \in \mathrm{Int}\overrightarrow{AB}$ if and only if $\angle PAB$ is acute.*

(*c*) *F lies in the interior of the ray opposite $\overrightarrow{AB}$ if and only if $\angle PAB$ is obtuse.*

Theorem 7.3. *Let $\triangle ABC$ be a triangle, and let F be the foot of the altitude from A to $\overline{BC}$.*

(*a*) *$F = B$ if and only if $\angle B$ is right, and $F = C$ if and only if $\angle C$ is right.*

(*b*) *$B * F * C$ if and only if $\angle B$ and $\angle C$ are both acute.*

(*c*) *$F * B * C$ if and only if $\angle B$ is obtuse, and $B * C * F$ if and only if $\angle C$ is obtuse.*

Corollary 7.4. *In any triangle, the altitude to the longest side always intersects the interior of that side.*

Corollary 7.5. *In a right triangle, the altitude to the hypotenuse always intersects the interior of the hypotenuse.*

Theorem 7.6 (Isosceles Triangle Altitude Theorem). *The altitude to the base of an isosceles triangle is also the median to the base, and is contained in the bisector of the angle opposite the base.*

Theorem 7.7 (Existence and Uniqueness of a Perpendicular Bisector). *Every segment has a unique perpendicular bisector.*

Theorem 7.8 (Perpendicular Bisector Theorem). *If $\overline{AB}$ is any segment, every point on the perpendicular bisector of $\overline{AB}$ is equidistant from A and B.*

Theorem 7.9 (Converse to the Perpendicular Bisector Theorem). *Suppose $\overline{AB}$ is a segment and P is a point that is equidistant from A and B. Then P lies on the perpendicular bisector of $\overline{AB}$.*

Theorem 7.10 (Reflection Across a Line). *Let ℓ be a line and let A be a point not on ℓ. Then there is a unique point A', called the **reflection of** A **across** ℓ, such that ℓ is the perpendicular bisector of $\overline{AA'}$.*

Lemma 7.12 (Properties of Closest Points). *Let P be a point and S be any set of points.*

(*a*) *If C is a closest point to P in S, then another point $C' \in S$ is also a closest point to P if and only if $PC' = PC$.*

(*b*) *If C is a point in S such that $PX > PC$ for every point $X \in S$ other than C, then C is the unique closest point to P in S.*

Theorem 7.13 (Closest Point on a Line). *Suppose ℓ is a line, P is a point not on ℓ, and F is the foot of the perpendicular from P to ℓ.*

(*a*) *F is the unique closest point to P on ℓ.*

(*b*) *If A and B are points on ℓ such that $F * A * B$, then $PB > PA$.*

Theorem 7.14 (Closest Point on a Segment). *Suppose $\overline{AB}$ is a segment and P is any point. Then there is a unique closest point to P in $\overline{AB}$.*

Theorem 7.15 (Angle Bisector Theorem). *Suppose $\angle AOB$ is a proper angle and P is a point on the bisector of $\angle AOB$. Then P is equidistant from $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$.*

Theorem 7.17 (Partial Converse to the Angle Bisector Theorem). *Suppose $\angle AOB$ is a proper angle. If P is a point in the interior of $\angle AOB$ that is equidistant from $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$, then P lies on the angle bisector of $\angle AOB$.*

Lemma 7.18. *Suppose ℓ is a line, and S is a segment, ray, or line that is parallel to ℓ. Then all points of S lie on the same side of ℓ.*

Theorem 7.19 (Alternate Interior Angles Theorem). *If two lines are cut by a transversal making a pair of congruent alternate interior angles, then they are parallel.*

Corollary 7.20 (Corresponding Angles Theorem). *If two lines are cut by a transversal making a pair of congruent corresponding angles, then they are parallel.*

Corollary 7.21 (Consecutive Interior Angles Theorem). *If two lines are cut by a transversal making a pair of supplementary consecutive interior angles, then they are parallel.*

Corollary 7.22 (Common Perpendicular Theorem). *If two distinct lines have a common perpendicular* (*i.e., a line that is perpendicular to both*)*, then they are parallel.*

Theorem 7.23 (Two-Point Equidistance Theorem). *Suppose ℓ and m are two distinct lines, and there exist two distinct points on ℓ that are on the same side of m and equidistant from m. Then $\ell \parallel m$.*

Theorem 7.25 (Existence of Parallels). *For every line ℓ and every point A that does not lie on ℓ, there exists a line m such that A lies on m and $m \parallel \ell$. It can be chosen so that ℓ and m have a common perpendicular that contains A.*

Some of Chapter 8 theorems

Theorem 8.4 (Vertex Criterion for Convexity). *A polygon $\mathfrak{P}$ is convex if and only if for every edge of $\mathfrak{P}$, the vertices of $\mathfrak{P}$ that are not on that edge all lie on the same side of the line containing the edge.*

Corollary 8.5. *Every triangle is a convex polygon.*

Theorem 8.6 (Angle Criterion for Convexity). *A polygon $\mathfrak{P}$ is convex if and only if for each vertex A_i of $\mathfrak{P}$, all the vertices of $\mathfrak{P}$ are contained in the interior of $\angle A_i$ except A_i itself and the two vertices consecutive with it.*

Theorem 8.7 (Semiparallel Criterion for Convexity). *A polygon is convex if and only if all pairs of nonadjacent edges are semiparallel.*

Theorem 8.9 (Polygon Splitting Theorem). *If $\mathfrak{P}$ is a convex polygon and $\overline{BC}$ is a chord of $\mathfrak{P}$, then the two subpolygons cut off by $\overline{BC}$ are both convex polygons.*

Chapter 9 theorems

Theorem 9.1. *Every rectangle is a parallelogram.*

Lemma 9.2. *In a convex quadrilateral, each pair of opposite vertices lies on opposite sides of the line through the other two vertices.*

Lemma 9.3. *Suppose $ABCD$ is a convex quadrilateral. Then $m\angle BAD = m\angle BAC + m\angle CAD$, with similar statements for the angles at the other vertices.*

Theorem 9.4 (Diagonal Criterion for Convex Quadrilaterals). *(a) If the diagonals of a quadrilateral intersect, then the quadrilateral is convex.*
(b) If a quadrilateral is convex, then its diagonals intersect at a point that is in the interiors of both diagonals and of the quadrilateral

Theorem 9.5 (Semiparallel Criterion for Convex Quadrilaterals). *If a quadrilateral has at least one pair of semiparallel sides, it is convex.*

Corollary 9.6. *Every trapezoid is a convex quadrilateral.*

Corollary 9.7. *Every parallelogram is a convex quadrilateral.*

Corollary 9.8. *Every rectangle is a convex quadrilateral.*

Lemma 9.9 (Cross Lemma). *Suppose $\overline{AC}$ and $\overline{BD}$ are noncollinear segments that have an interior point in common. Then ABCD is a convex quadrilateral.*

Lemma 9.10 (Trapezoid Lemma). *Suppose A, B, C, and D are four distinct points such that $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \cap \overline{BC} = \varnothing$. Then ABCD is a trapezoid.*

Lemma 9.11 (Parallelogram Lemma). *Suppose A, B, C, and D are four distinct points such that $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$. Then ABCD is a parallelogram.*

Theorem 9.12 (SASAS congruence). *Suppose ABCD and EFGH are convex quadrilaterals such that $\overline{AB} \cong \overline{EF}$, $\overline{BC} \cong \overline{FG}$, $\overline{CD} \cong \overline{GH}$, $\angle B \cong \angle F$, and $\angle C \cong \angle G$. Then $ABCD \cong EFGH$.*

Theorem 9.13 (AASAS Congruence). *Suppose ABCD and EFGH are convex quadrilaterals such that $\angle A \cong \angle E$, $\angle B \cong \angle F$, $\angle C \cong \angle G$, $\overline{BC} \cong \overline{FG}$, and $\overline{CD} \cong \overline{GH}$. Then $ABCD \cong EFGH$.*

Theorem 9.14 (Copying a Quadrilateral). *Suppose ABCD is a convex quadrilateral, and $\overline{EF}$ is a segment congruent to $\overline{AB}$. On either side of $\overleftrightarrow{EF}$, there are distinct points G and H such that $EFGH \cong ABCD$.*

Theorem 9.15. *A convex quadrilateral with two pairs of congruent opposite angles is a parallelogram.*

Corollary 9.16. *Every equiangular quadrilateral is a parallelogram.*

Theorem 9.17. *A quadrilateral with two pairs of congruent opposite sides is a parallelogram.*

Corollary 9.18. *Every rhombus is a parallelogram.*

Theorem 9.19. *Suppose ABCD is a quadrilateral.*

(*a*) *If its diagonals bisect each other, then ABCD is a parallelogram.*
(*b*) *If its diagonals are congruent and bisect each other, then ABCD is equiangular.*
(*c*) *If its diagonals are perpendicular bisectors of each other, then ABCD is a rhombus.*
(*d*) *If its diagonals are congruent perpendicular bisectors of each other, then ABCD is a regular quadrilateral.*

Corollary 9.20. *There exists a regular quadrilateral.*