Pick's Theorem

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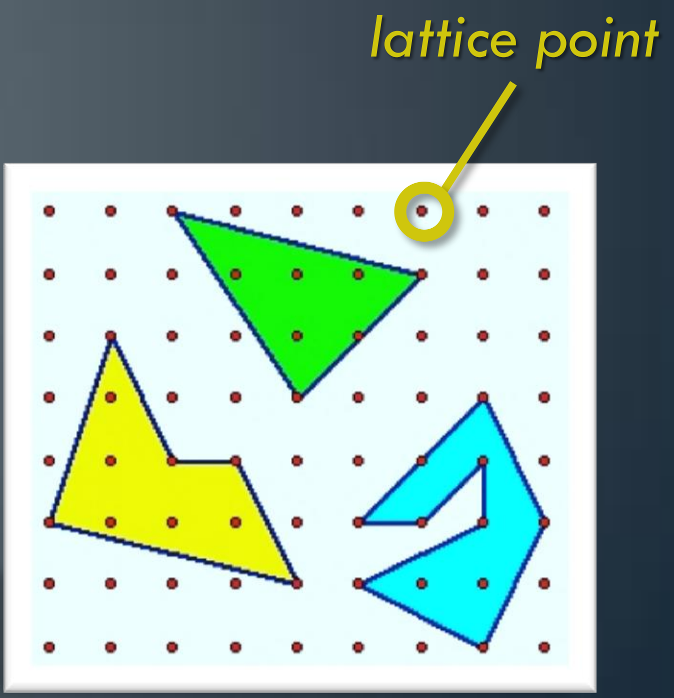
Finding the Area of Lattice Polygons

- Pick's Formula provides a simple formula for the area of any *lattice polygon*

A lattice polygon is a polygon embedded on a grid of equidistant points, or lattice, such that each of the polygon's vertices lie on grid points.



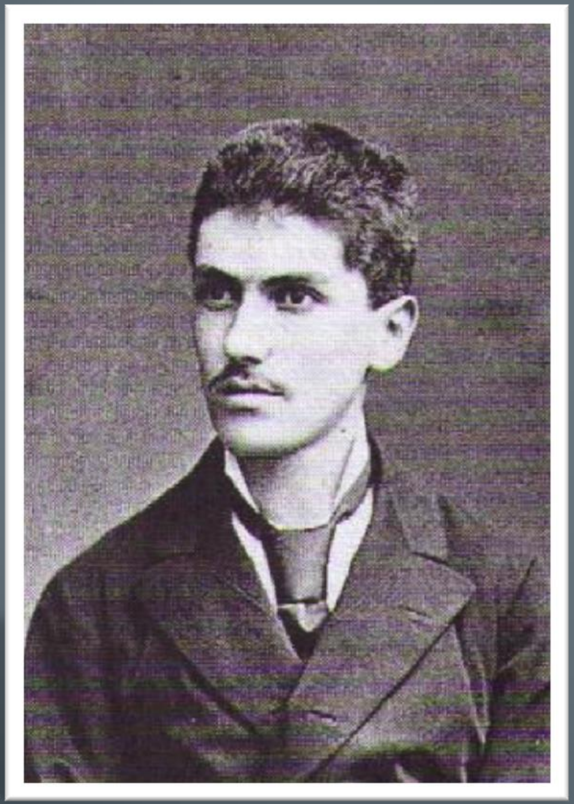
Finding the Area of Lattice Polygons

- The formula involves simply counting lattice points:
  - $b =$ number of lattice points on the boundary of a polygon
  - $i =$ number of lattice points on the interior of a polygon

- Given a lattice polygon P, the area of P according to Pick's Formula is

$$\boldsymbol{A(P) = \frac{b}{2} + i - 1}$$

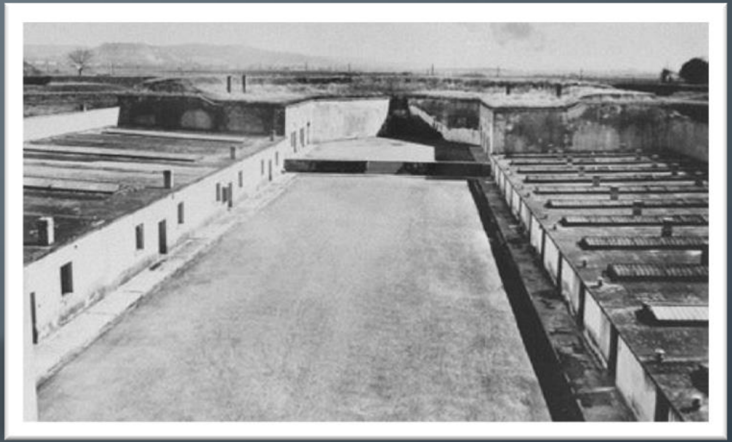
Georg Alexander Pick



- Born in Vienna, 1859
- Attended University of Vienna at 16; published first math paper at 17
- Dean of Philosophy at University of Prague
- Studied wide range of topics:
  - Pick Matrices
  - Pick-Nevalinna interpolation
  - Schwarz-Pick lemma
- Stated area theorem in 1899

Georg Alexander Pick

- Tragically, Pick was killed in the Holocaust after the Nazis invaded Czechoslovakia in 1939
  - He died in 1942, at 82 years old, in Theresienstadt concentration camp
- His area formula didn't become famous until Hugo Steinhaus included it in his famous book, "Mathematical Snapshots"



Proof

Part 1: Rectangles

Calculation:

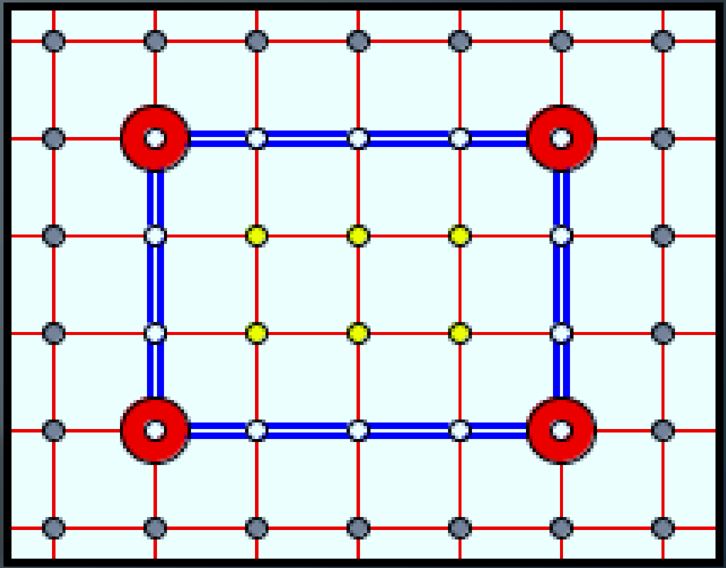
$$A(P) = \frac{b}{2} + i - 1$$

$$A(P) = \frac{14}{2} + 6 - 1$$

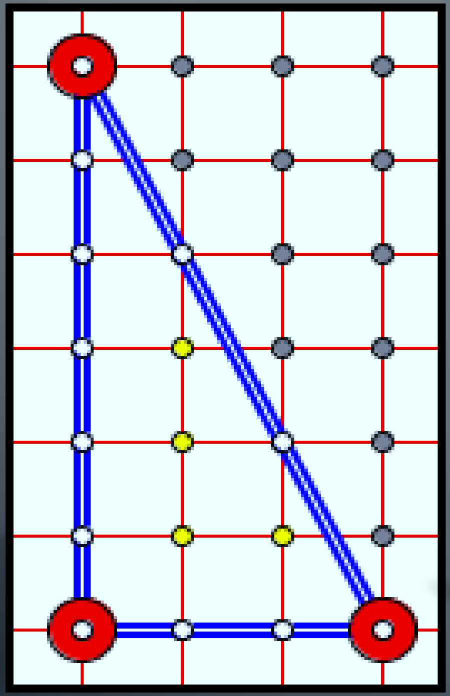
$$A(P) = 12$$

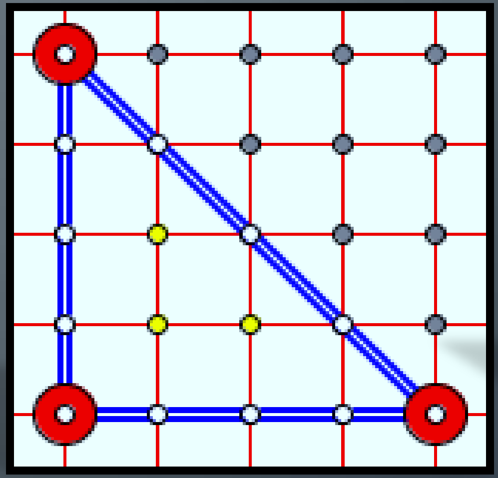
Theorem used in proof:

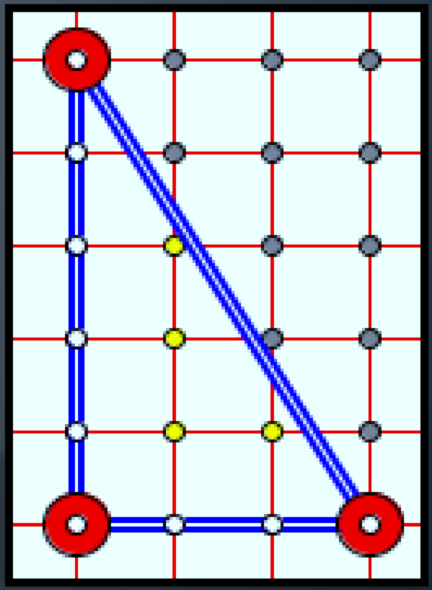
Lemma 11.5



Part 2: Right Triangles







Part 2: Right Triangles

Calculation:

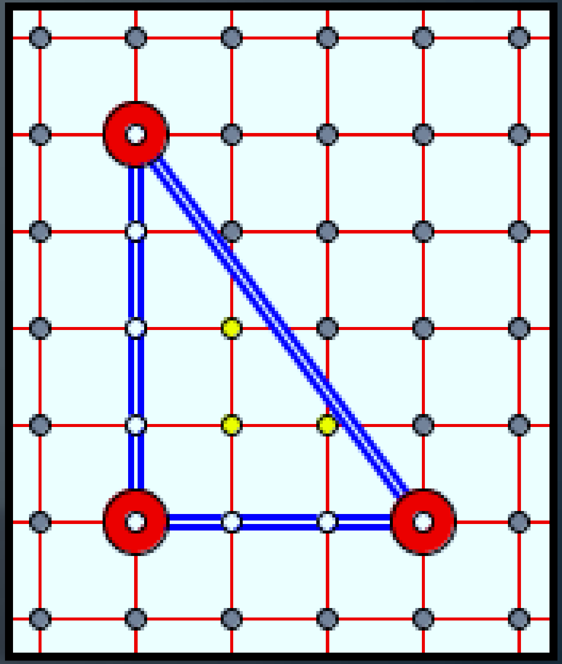
$$A(P) = \frac{b}{2} + i - 1$$

$$A(P) = \frac{8}{2} + 3 - 1$$

$$A(P) = 6$$

Theorem used in proof:

Lemma 11.9



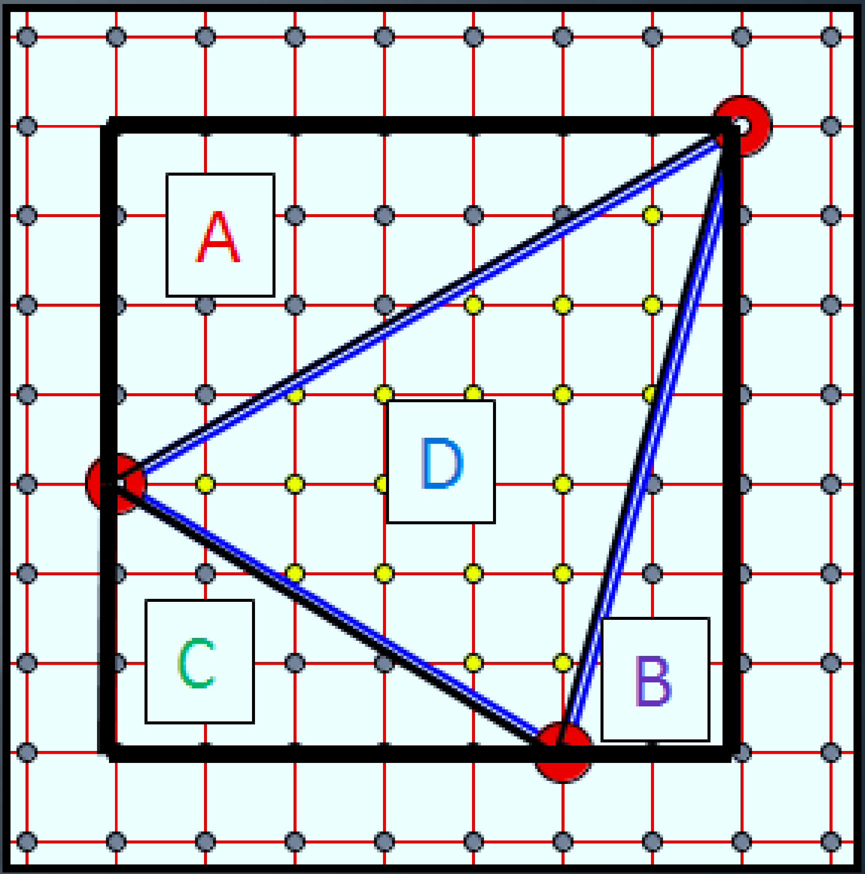
Part 3: General Triangles

Theorem used in proof:

Lemma 11.2

$$A(\boldsymbol{R}) = A(A) + A(B) + A(C) + A(\boldsymbol{D})$$

$$A(\boldsymbol{D}) = A(\boldsymbol{R}) - [A(A) + A(B) + A(C)]$$



Part 3: General Triangles

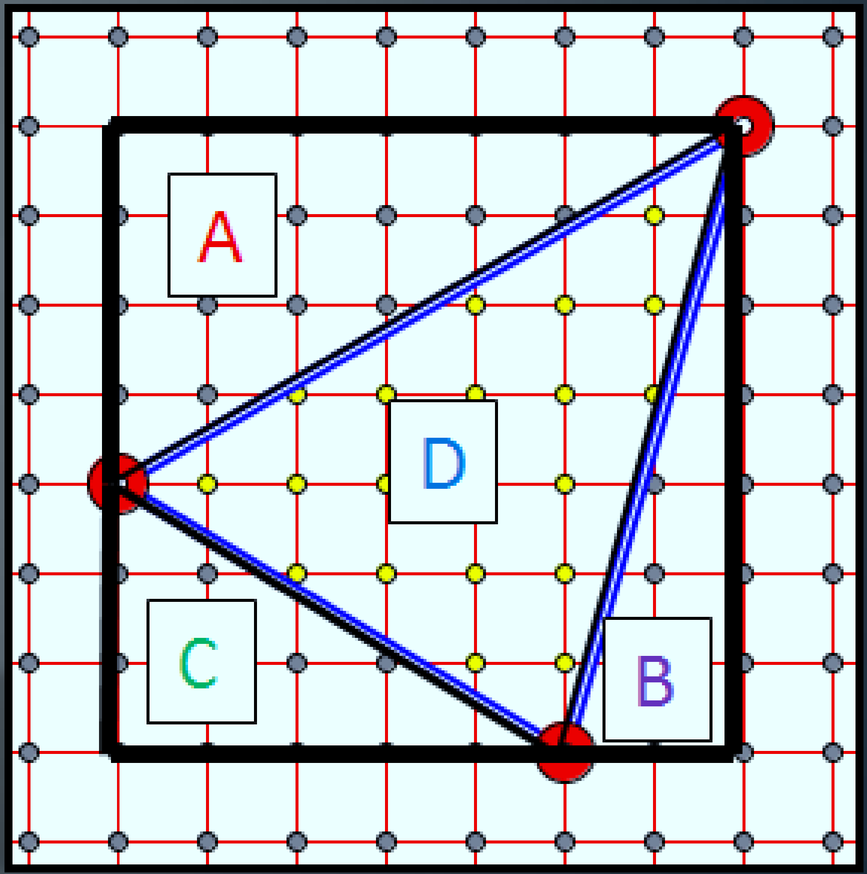
$$A(R) = \frac{B_R}{2} + I_R - 1$$

$$A(A) = \frac{B_A}{2} + I_A - 1$$

$$A(B) = \frac{B_B}{2} + I_B - 1$$

$$A(C) = \frac{B_C}{2} + I_C - 1$$

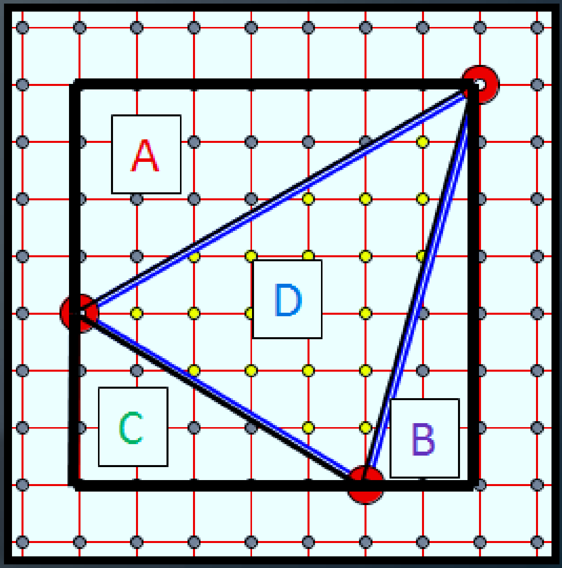
$$A(D) = \frac{B_D}{2} + I_D - 1$$



Part 3: General Triangles

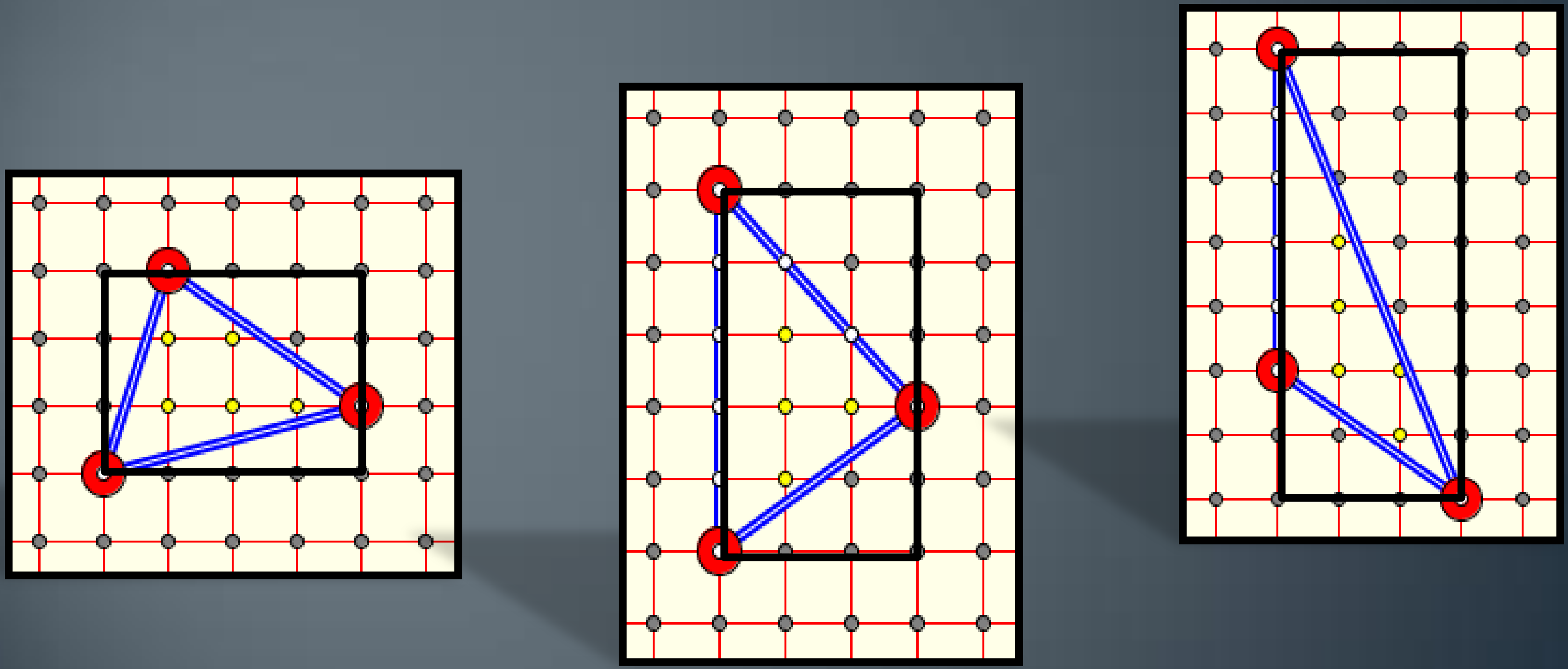
1. $A(D) = I_R - I_A - I_B - I_C + \frac{B_R - B_A - B_B - B_C}{2} + 2$
2. $B_R = B_A + B_B + B_C - B_D$
3. $I_R = I_A + I_B + I_C + I_D + (B_A + B_B + B_C - B_R) - 3$
4. $I_R = I_A + I_B + I_C + I_D + B_D - 3$

$$\boldsymbol{A(D) = I_D + \frac{B_D}{2} - 1}$$



Part 3: General Triangles

What other possible cases exist?



Part 4: Overview

Pick's Theorem is **additive.**

$$A\,(P) = A\,(P_1) + A\,(P_2) = (I_1 + B_1/2 - 1) + (I_2 + B_2/2 - 1)$$

$$I + \frac{B}{2} - 1 = I_1 + I_2 + d - 2 + \frac{B_1 + B_2 - 2(d-2) - 2}{2} - 1$$

$$= I_1 + I_2 + d - 2 + \frac{B_1}{2} + \frac{B_2}{2} - d + 2 - 1 - 1$$

$$= I_1 + I_2 + \frac{B_1}{2} + \frac{B_2}{2} - 2$$

$$= \left(I_1 + \frac{B_1}{2} - 1\right) + \left(I_2 + \frac{B_2}{2} - 1\right)$$

$$= A\,(P_1) + A\,(P_2) = A\,(P)$$

Part 5: Inner Diagonal

Calculation:

$$A(P) = \frac{b}{2} + i - 1$$

$$A(P) = \frac{18}{2} + 18 - 1$$

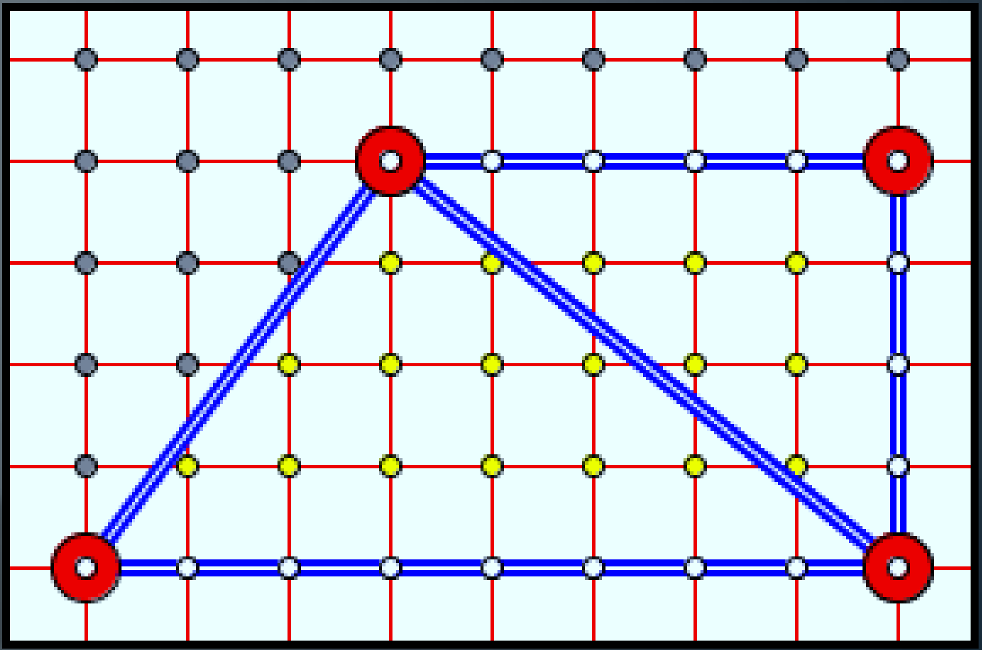
$$A(P) = 26$$

Theorem used in proof:

Theorem 10.19

Theorem 8.7

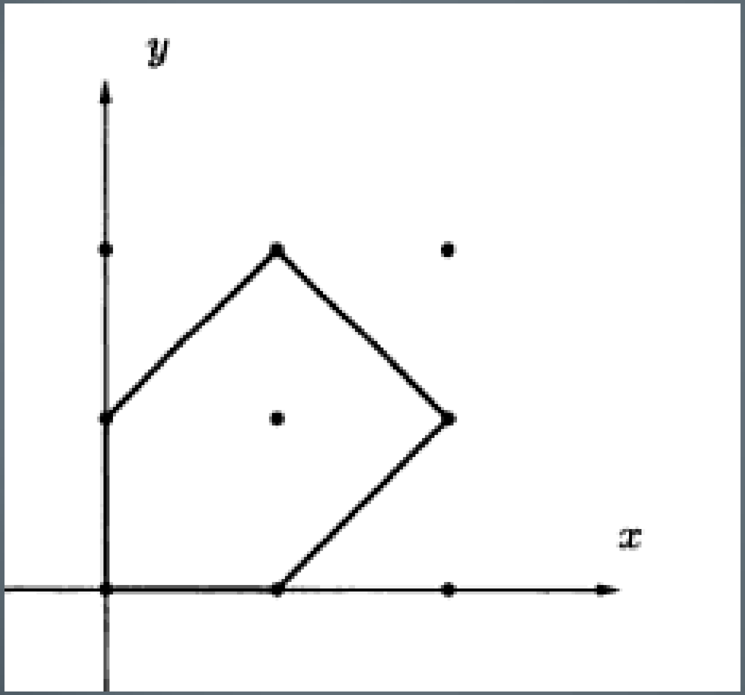
Theorem 10.11



An Application

What is the minimum area of a convex lattice pentagon?

5/2

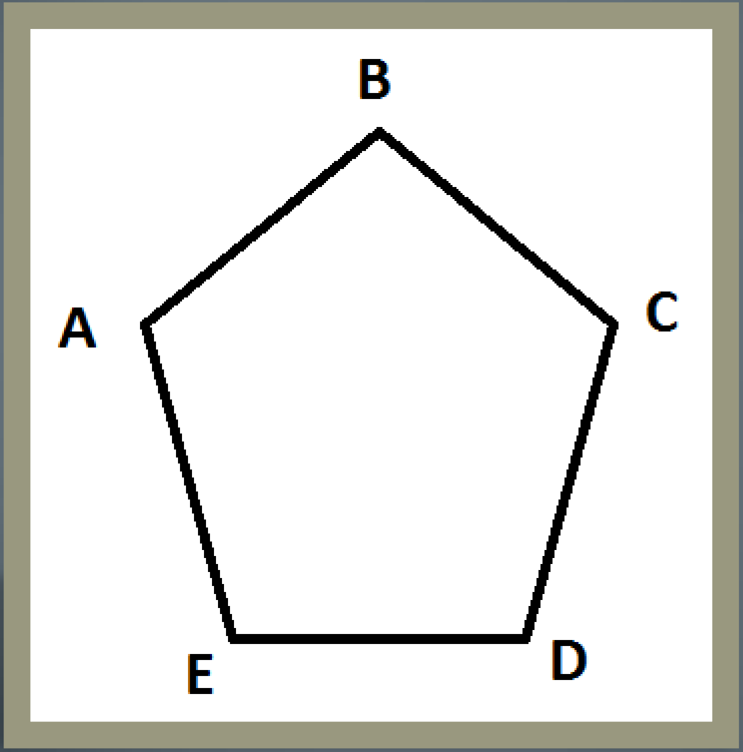


- This pentagon has vertices: (0,0), (0,1), (1,2), (2,1) and (1,0)
- I = 1 and B = 5
- A = 1 + 5/2 - 1 = 5/2

Properties of evens and odds

- Odd + Odd = Even
  - $(2m + 1) + (2n + 1) = 2m + 2n + 2 = 2(m + n + 1)$
- Even + Even = Even
  - $2m + 2n = 2(m + n)$
- Odd + Even = Odd
  - $2m + 1 + 2n = 2(m + n) + 1$

Pigeonhole Principle and Parity Classes



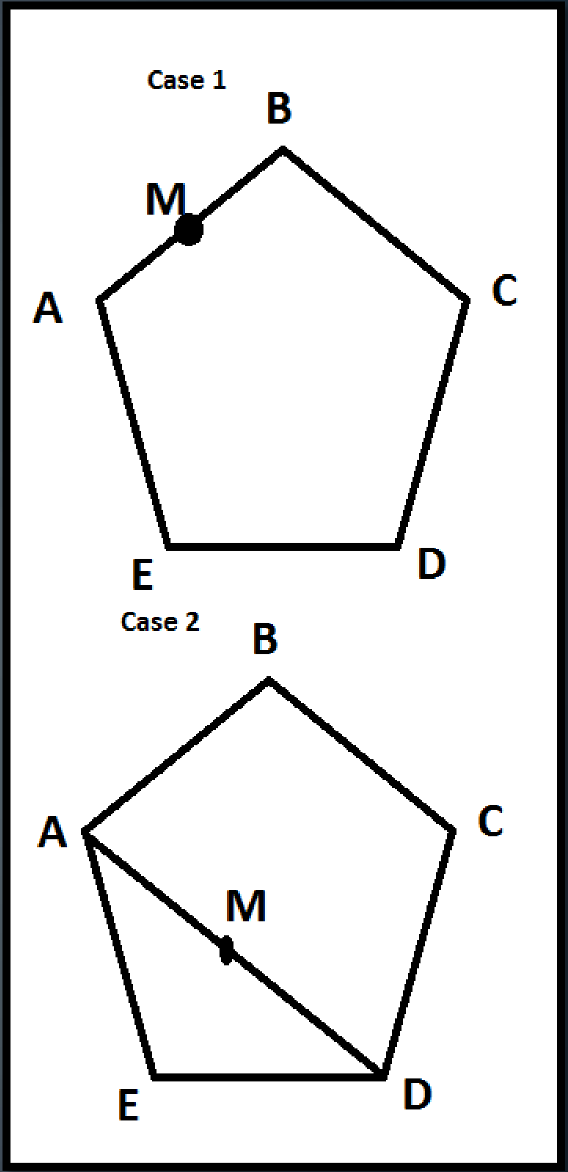
- A pentagon has five vertices each with integer coordinates
- There are four “parity classes”
- (odd, odd), (even, even), (even, odd), and (odd, even)
- Two vertices must be in the same class

Midpoint Formula

- $Q = (x_Q, y_Q)$ and $R = (x_R, y_R)$
- $M = \left(\frac{x_Q + x_R}{2}, \frac{y_Q + y_R}{2}\right)$
- M is a lattice point with integer coordinates if $x_Q + x_R$ and $y_Q + y_R$ are even
- Two vertices in the same parity class have a midpoint with integer coordinates
- One midpoint of ABCDE has integer coordinates

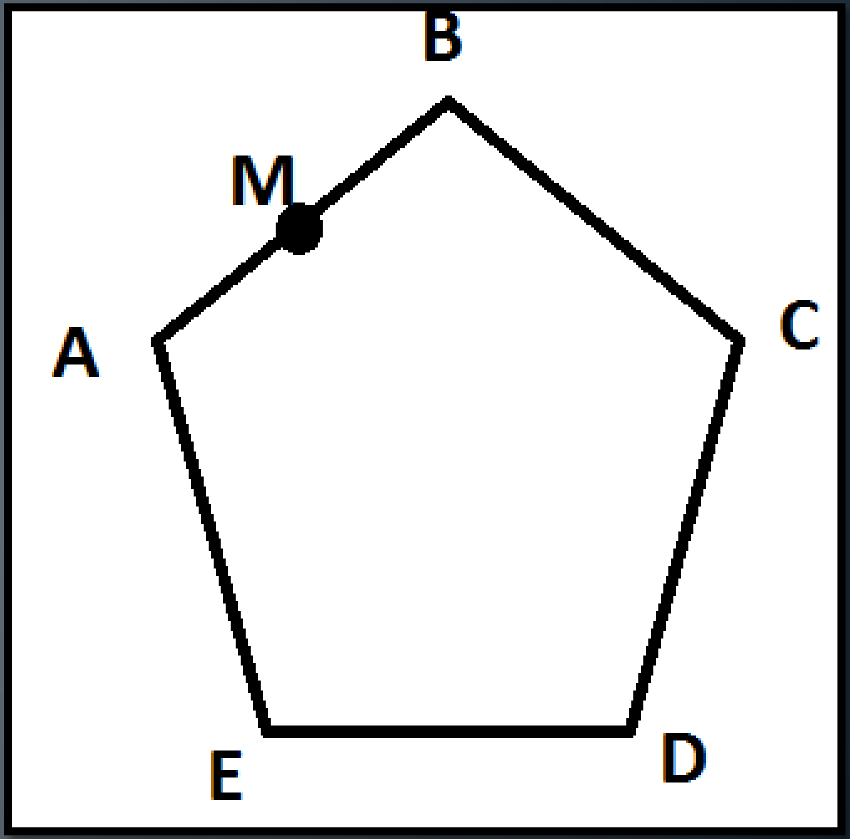
Cases 1 and 2

- We can say without loss of generality A is one of vertices
- Two Cases
- Same parity as adjacent vertex (B or E) (Case 1)
- Same parity as another vertex (C or D) (Case 2)

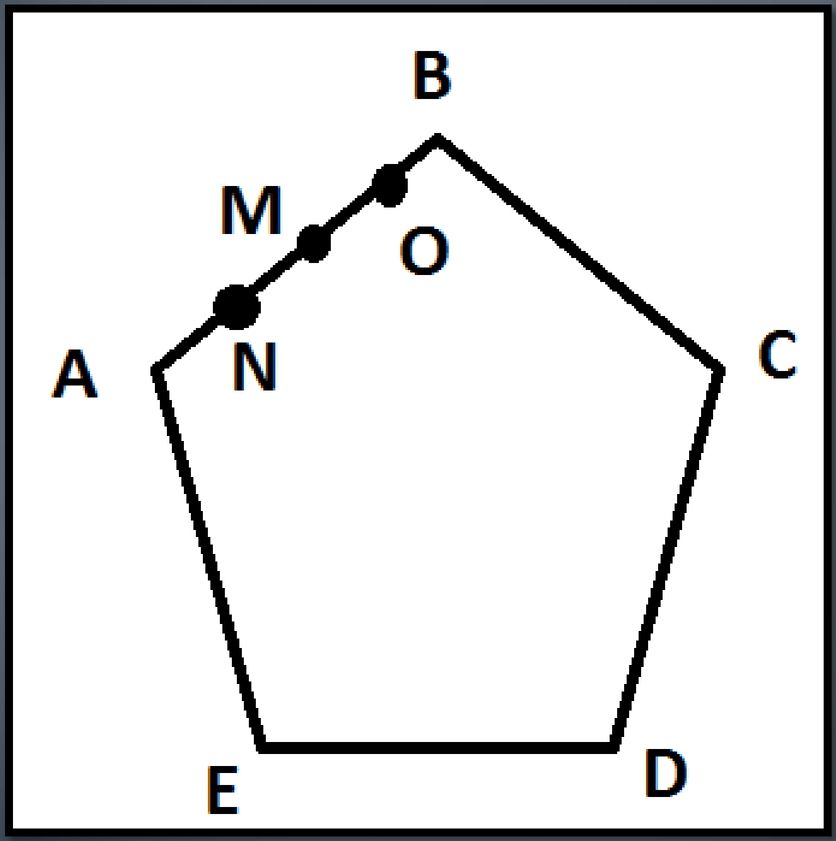


Case 1

- Can assume the vertex is B
- Let M be the midpoint of $\overline{AB}$
- There are two subcases
  - A: M is of the same parity of A and B
  - B: M is of different parity of A and B



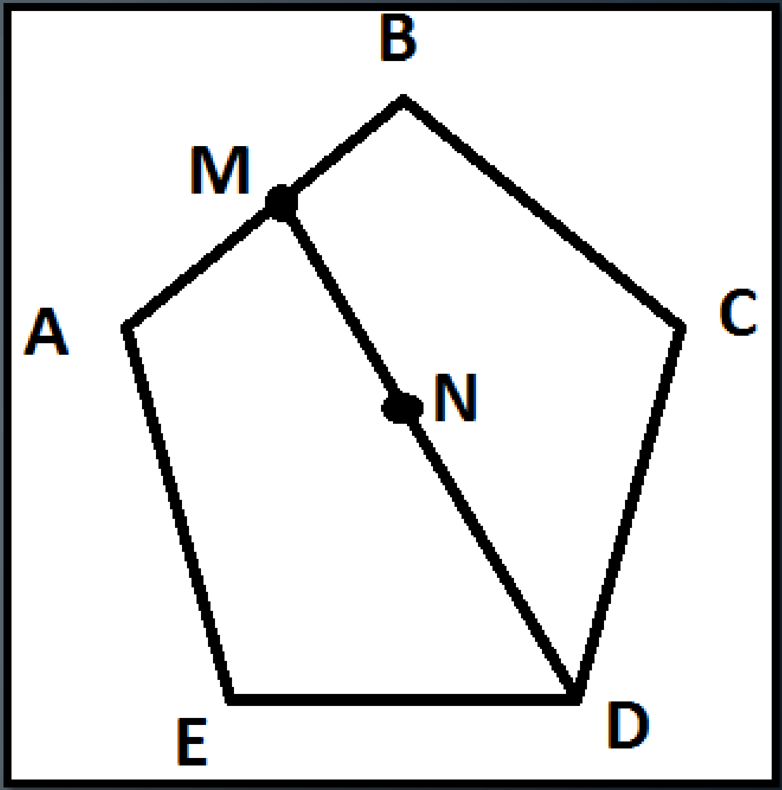
Case 1A



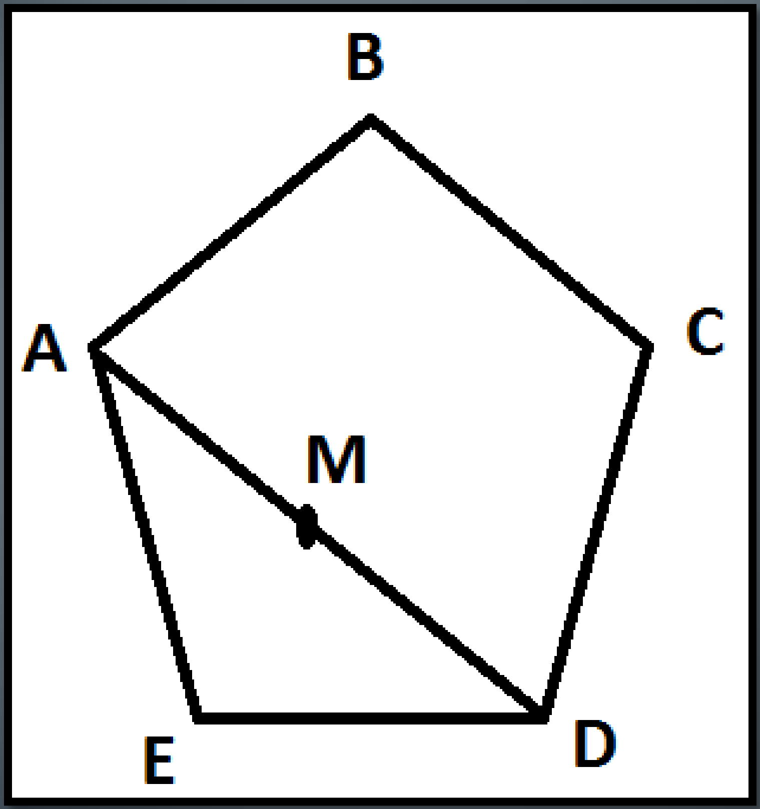
- In this case M, A, and B all have the same parity
- Hence the midpoints of $\overline{MA}$ and $\overline{MB}$ are also on the boundary
- With 3 additional boundary points
- $I \geq 0$, $B \geq 8$
- $A = 0 + 8/2 - 1 = 3$

Case 1B

- In this case M has different parity then A and B
- By the pigeonhole principle A has the same parity as C, D, or E
- Rename vertex F
- Midpoint of $\overline{MF}$ is in the interior because ABCDE is convex
- With a additional boundary point and an interior point
- $I \geq 1$, $B \geq 6$
- $A = 1 + 6/2 - 1 = 3$



Case 2



- The two vertices with the same parity are not adjacent
- Midpoint is the in the interior
- Reasoning same as 1b
- One interior point
- $I \geq 1$, $B \geq 5$
- $A = 1 + 5/2 - 1 = 5/2$

Summary

Case	Min B	Min I	Min Area
1a	8	0	3
1b	6	1	3
2	5	1	5/2

Fin