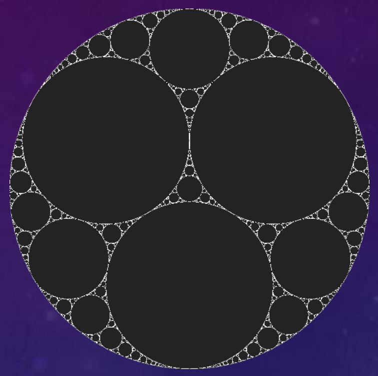

APOLLONIAN GASKET AND THE DESCARTES' THEOREM

THE FRACTAL THAT INSPIRES MATHEMATICAL POETRY

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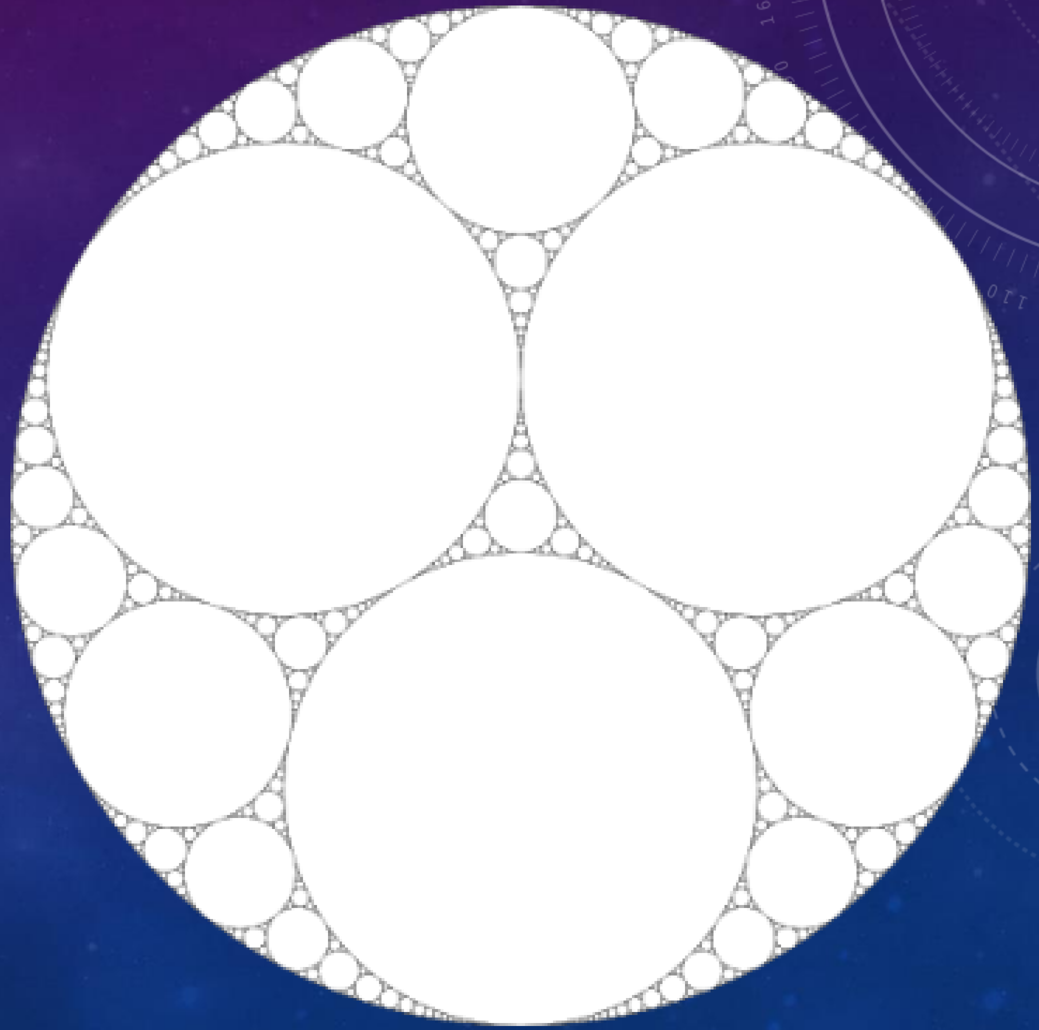
APOLLONIAN GASKET

- An Apollonian Gasket is a group of three circles in which each circle is tangent to the other two.
- It is named after the Greek mathematician Apollonius of Perga.

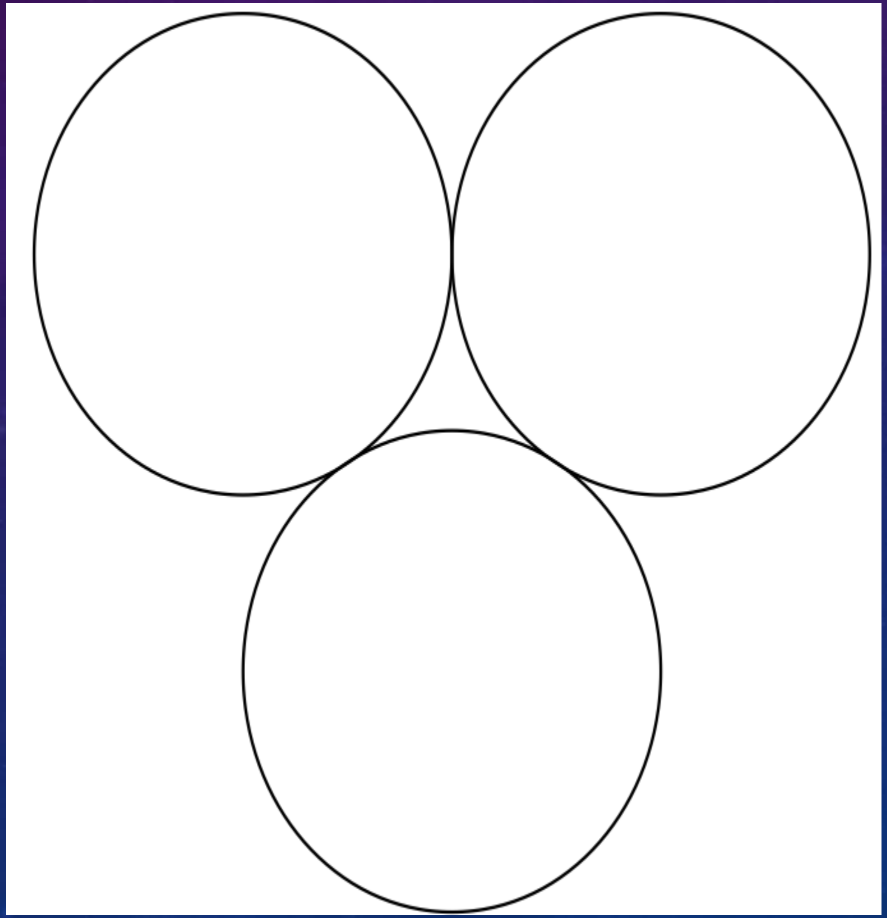


IMPORTANT TERMS

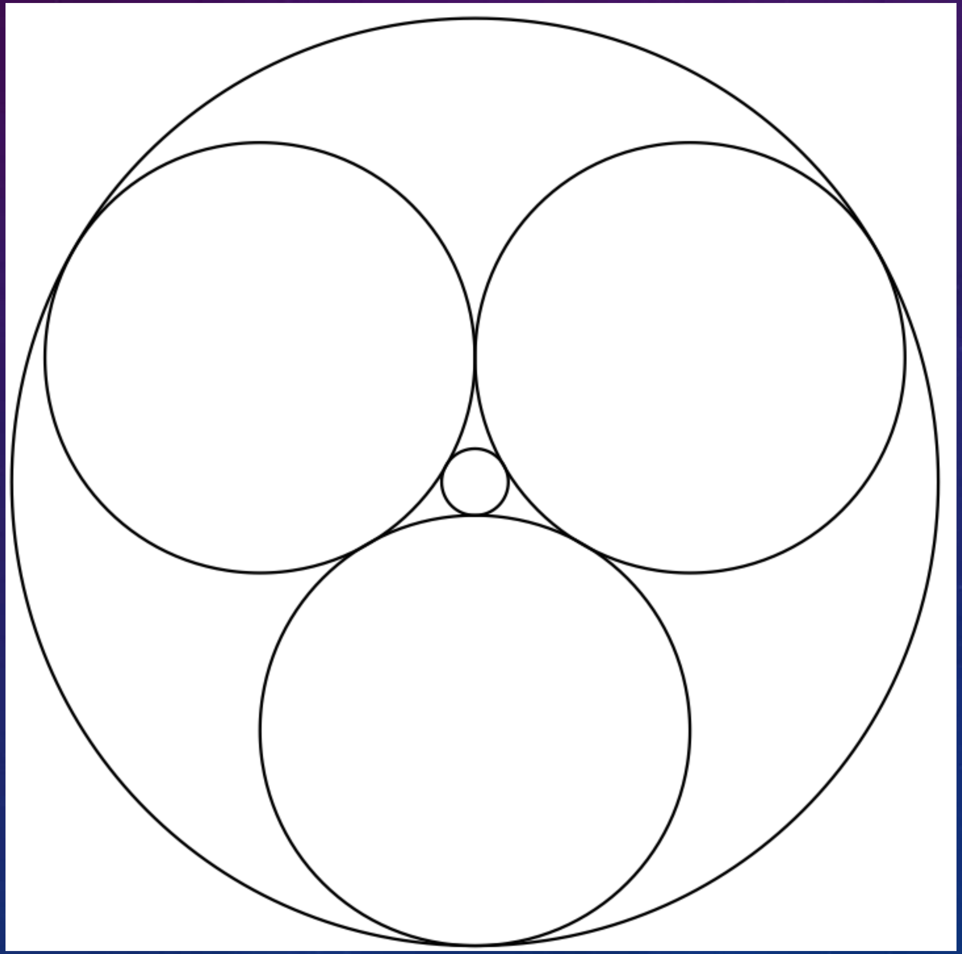
- Apollonian Gasket: one of several names for a series of circles, they are inside one large circle and tangent to others nearby. Also called Kissing circles.
- Radius of circle: distance from the center point to the edge of a circle.
- Curvature of circle: inverse of the radius, +- 1/r,+ when dealing with the outer curvature,- for the inner.
- Tangent: when two circles have a point in common. They have same slope at that point and no intersection.
- Descartes’ theorem: the formula we use to calculate all circles’ sizes.

STEPS OF CONSTRUCTION

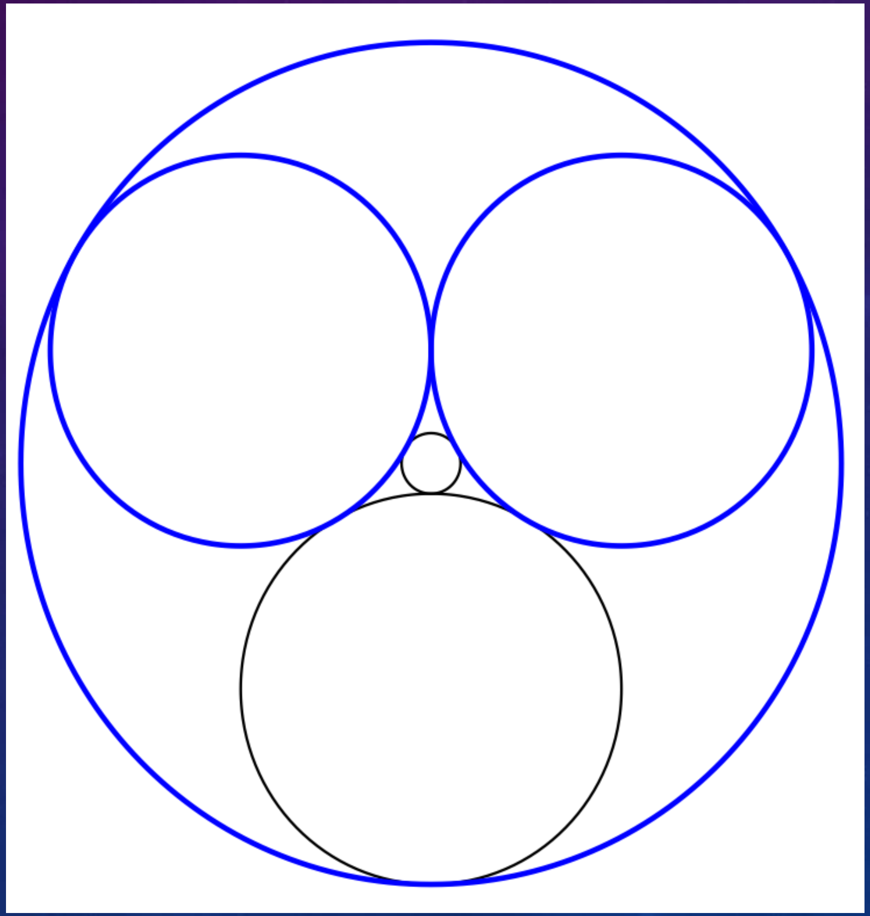
1. Make three small circles,c_1,c_2,c_3. each one is tangent to other two.



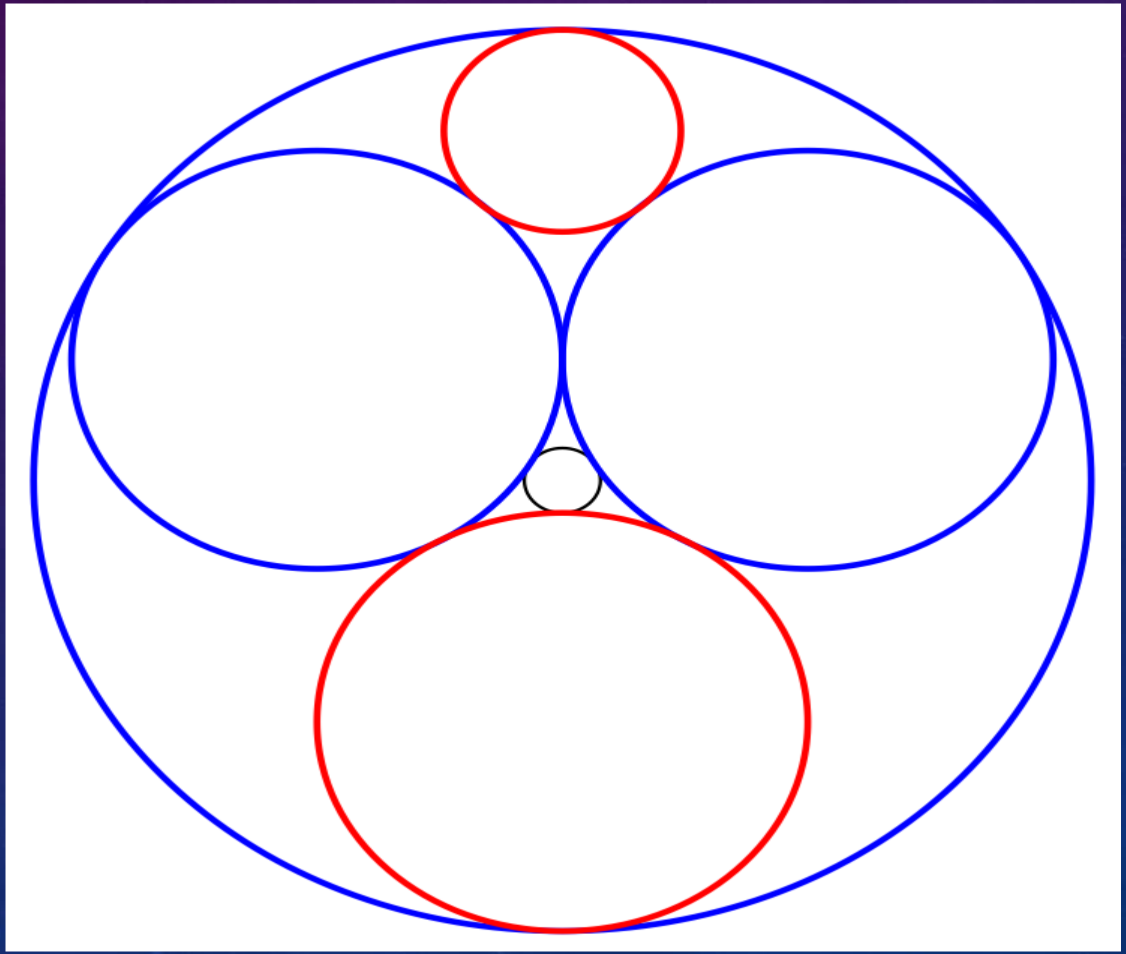
2. Then by Descartes' theorem, there exists two circles C_4,C_5, tangent to all three of C_1,C_2,C_3.



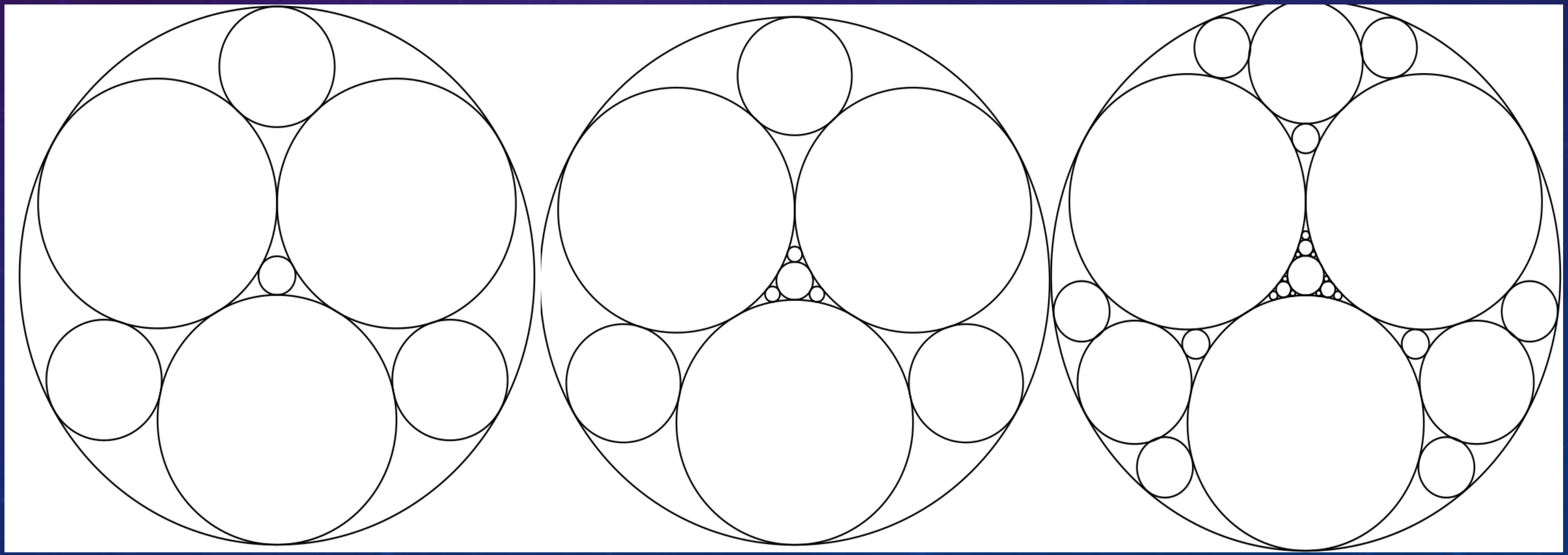
3. Those two circles are called Apollonian Circles, so C_4 is the outer circle, C_5 is the small one in the center of original three.



4. Since C_4 is tangent to C_1 and C_2, we can apply the same process starting from those three circles, and the same goes for C_5. and a new circle.



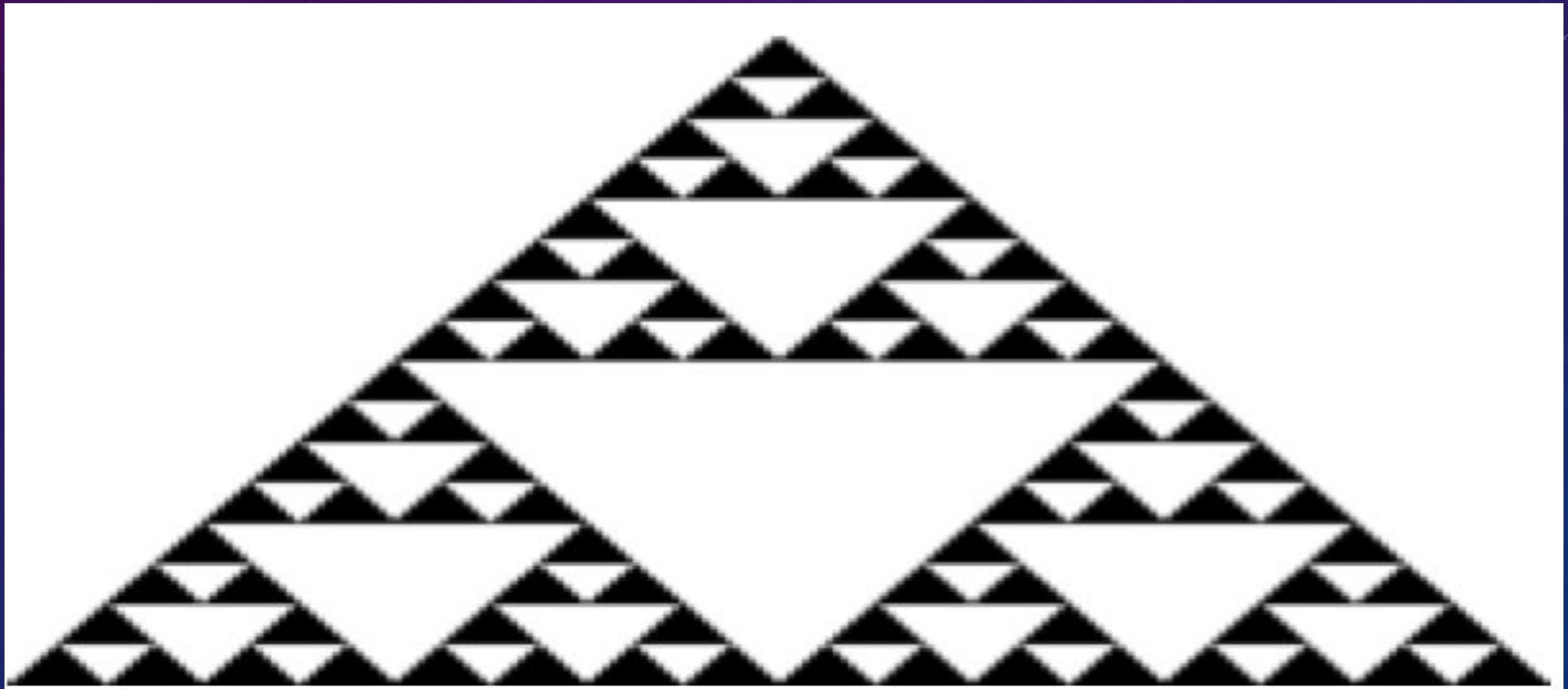
5. In fact, we can do this with any set of three circles consisting one of circle from C_4,C_5 and two from C_1,C_2,C_3. this give us a pattern where beginning from 3 tangent circles. In general, we can add $2*3^n$ new circles at stage n, and giving a total of $3^{(N+1)}+2$ circles after stage n. the infinite set of circles is an Apollonian gasket.



MORE ABOUT CURVATURE

- To calculate the curvature of the circle that is tangent to any set of three tangent circles, we apply Descartes' Theorem. The curvature has three possible results: Negative curvature means that all other circles are internally tangent to that circle, like C_4. Positive curvature means that all other circles are externally tangent to that circle, like C_5. Zero curvature means there is a line.

SIERPINSKI TRIANGLE



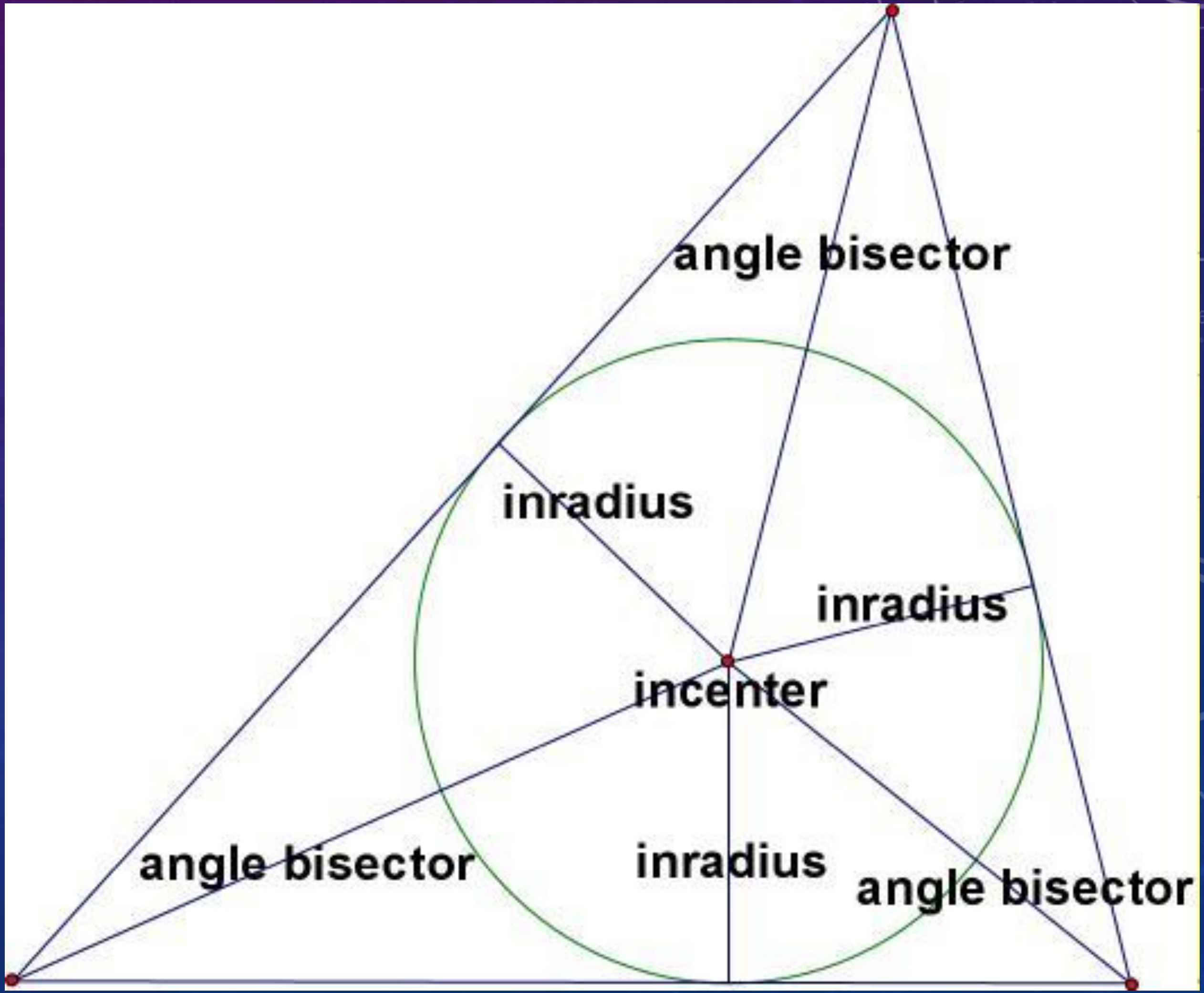
- First step of creating a Sierpinski triangle is having a large equilateral triangle
- Second step is pointing out the midpoint of each side of equilateral triangle, then connect those midpoints to make a new triangle inside the bigger one. So far, we will have four equal triangles inside the bigger one, one of which will be inverted. Leave the upside down triangle alone, and apply the second step for other three triangles to make more small equilateral triangles.
- This is continued indefinitely to create a fractal pattern. Consider the state after the first iteration of this process, when there are three small equilateral triangles with one upside down in their center. This is quite similar to the situation when you have three mutually tangent circles in place of the upright triangles, and there is one circle in the center tangent to all three of them.

DESCARTES' THEOREM

- Named after the French philosopher, mathematician, and writer René Descartes.
- Establishes a relationship between four "kissing", or mutually tangent circles.
- Descartes provided his discussion of the problem in 1693, but his proof was incomplete.

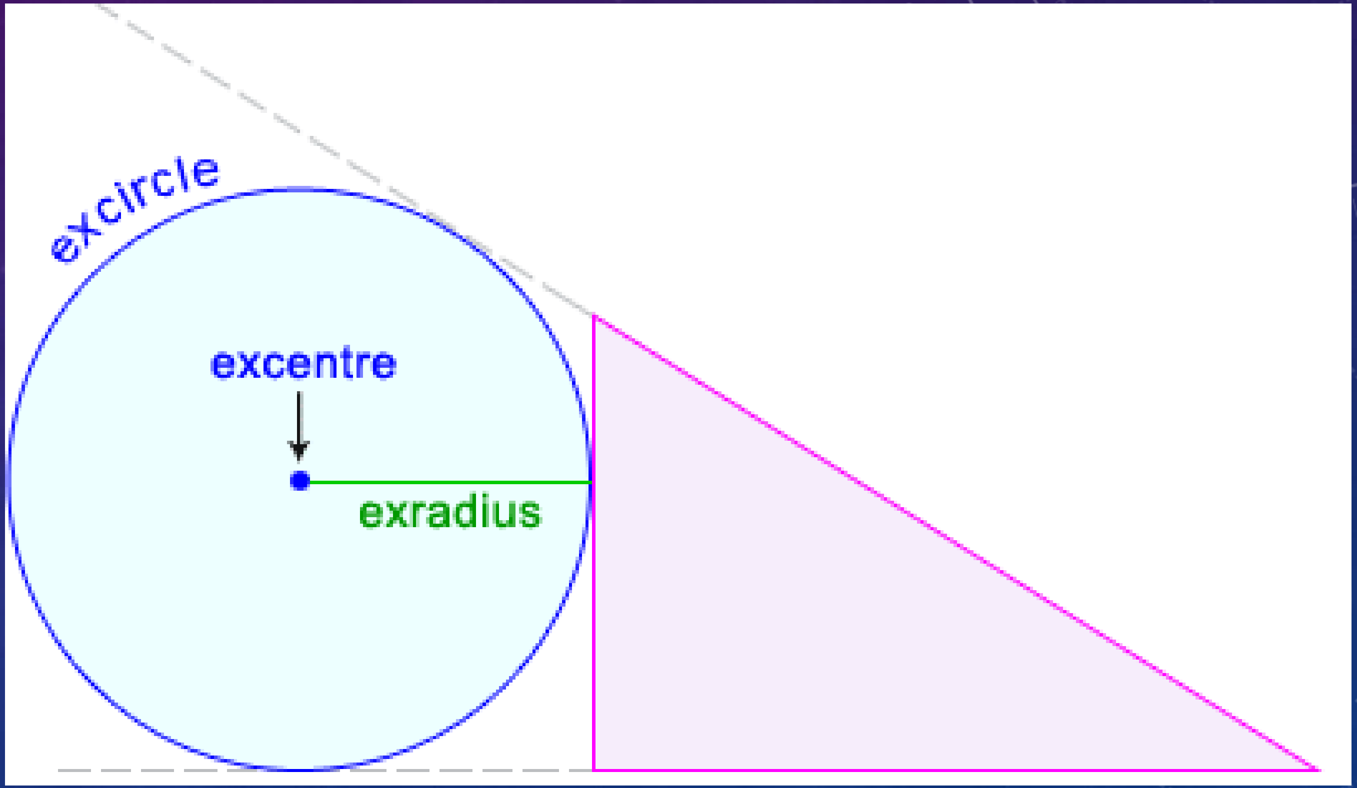
DEFINITIONS

- Incircle - Given any triangle, the incircle is the largest circle contained in the triangle which is tangent to all three of the sides.
- Incenter – The center of the incircle.
- Inradius – The radius of the incircle.



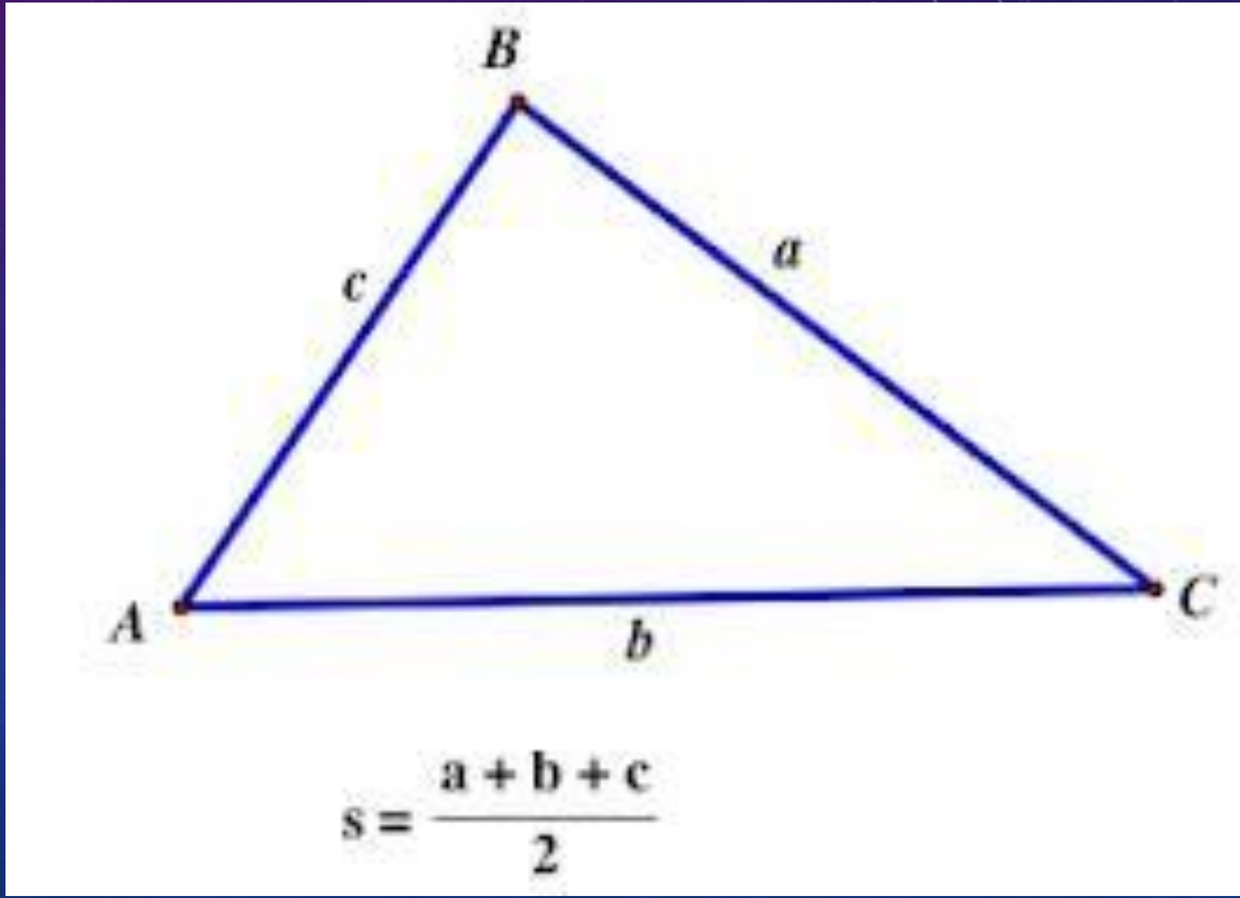
DEFINITIONS CONTINUED

- Excircle - Given a triangle, extend two sides in the direction opposite their common vertex. The circle tangent to both of these lines and to the third side of the triangle is called an excircle.
- Exradius – The radius of an excircle.



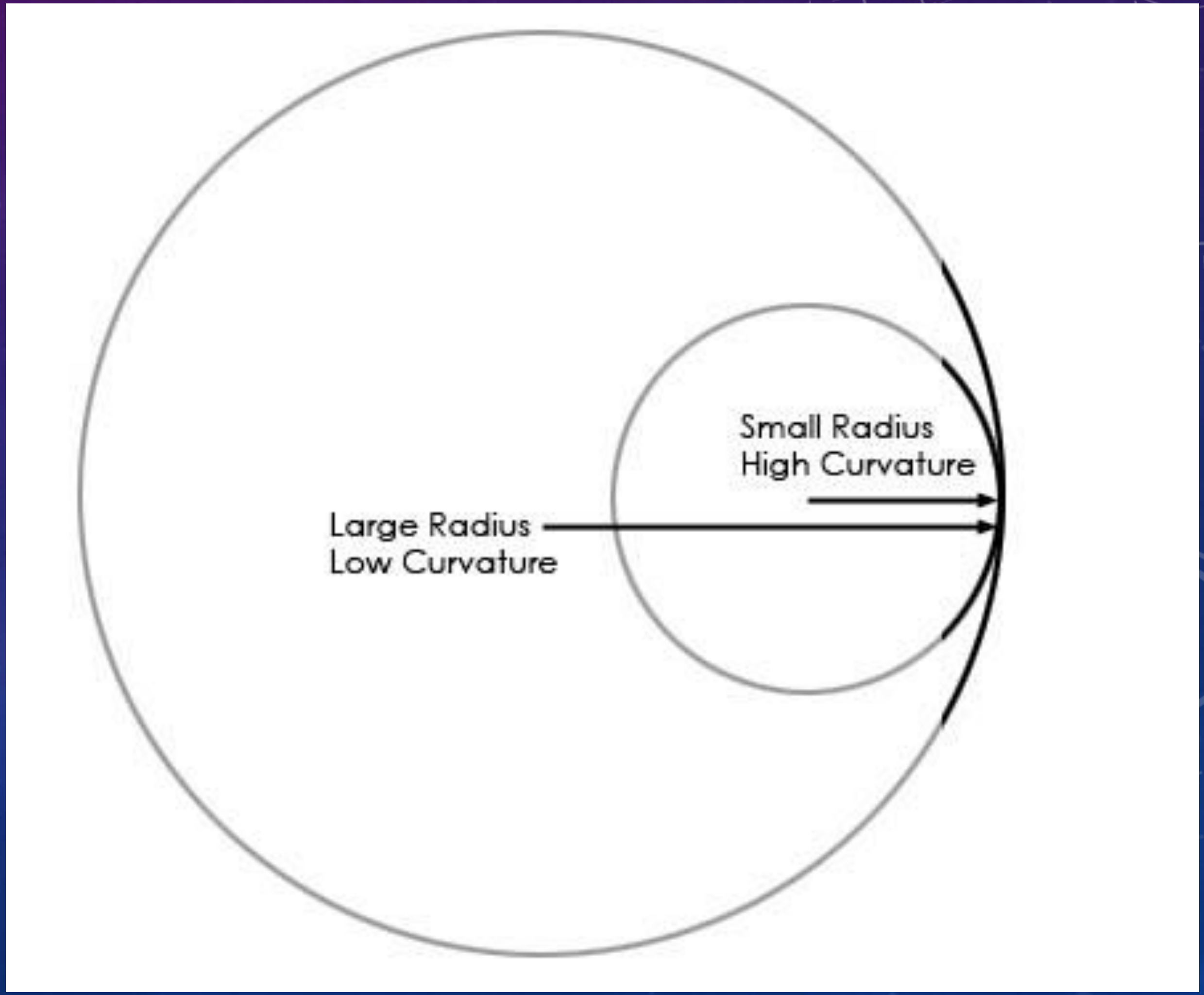
DEFINITIONS CONTINUED

- Semiperimeter - Half of the perimeter of a polygon (we will be using the semiperimeter of a triangle).



DEFINITIONS CONTINUED

- Curvature (or “bends”) of a circle – The reciprocal of the radius of that circle (1/r).



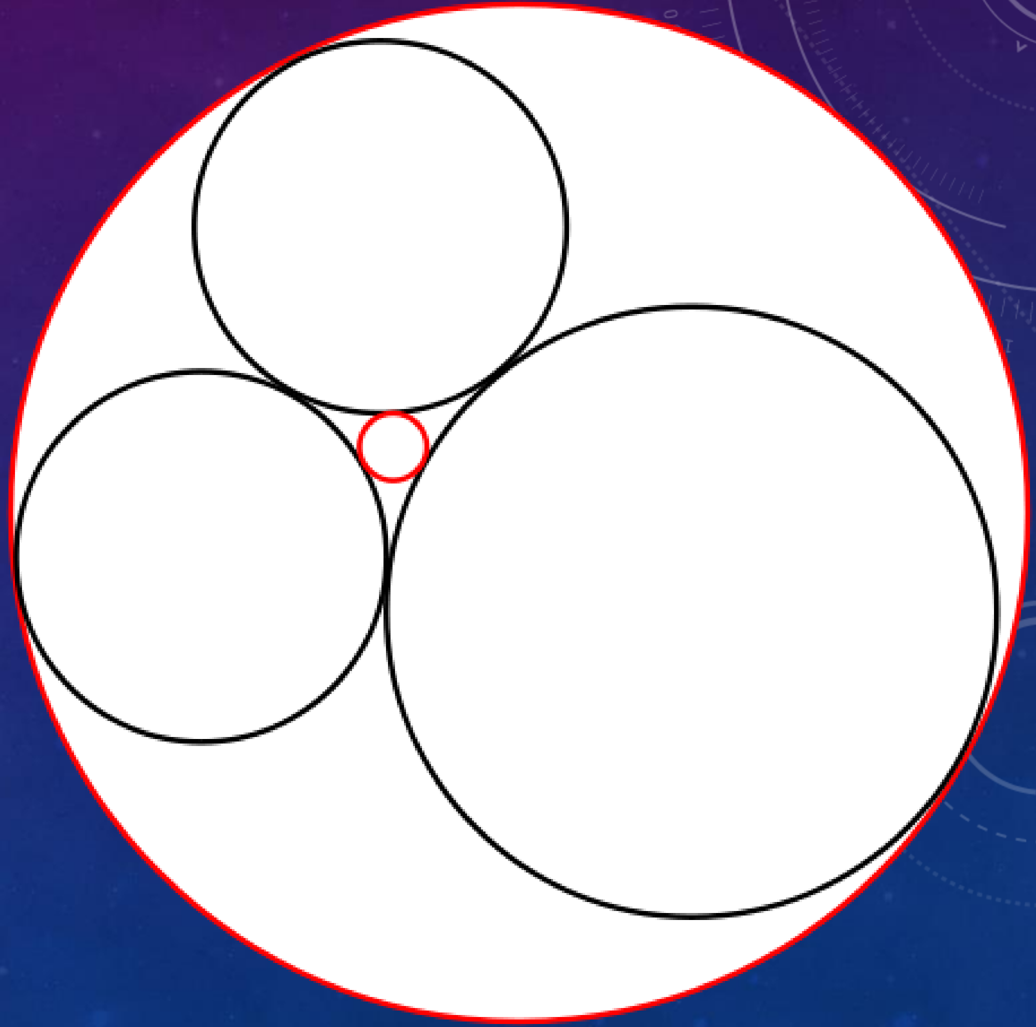
FORMULATION OF THE DESCARTES' THEOREM

- In a Descartes configuration of four mutually tangent circles, the following can be said about their curvatures (with b_1, b_2, b_3, and b_4 representing their respective curvatures):

$$2\,(b_1^2 + b_2^2 + b_3^2 + b_4^2) = (b_1 + b_2 + b_3 + b_4)^2$$

or

$$2 \sum b_i^2 = \left(\sum b_i\right)^2$$

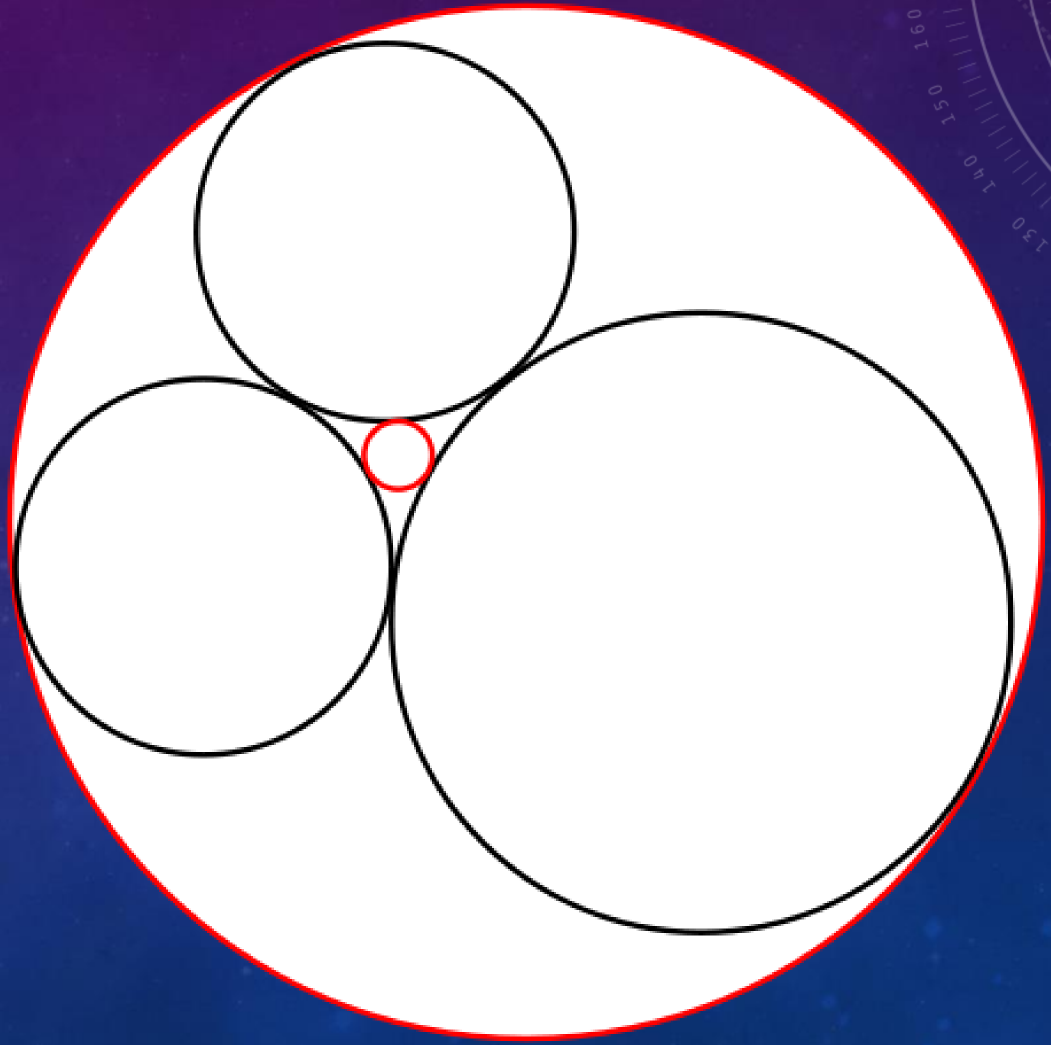


CAN YOU THINK OF ANY OTHER CASES?

$$2\,(b_1^2 + b_2^2 + b_3^2 + b_4^2) = (b_1 + b_2 + b_3 + b_4)^2$$

or

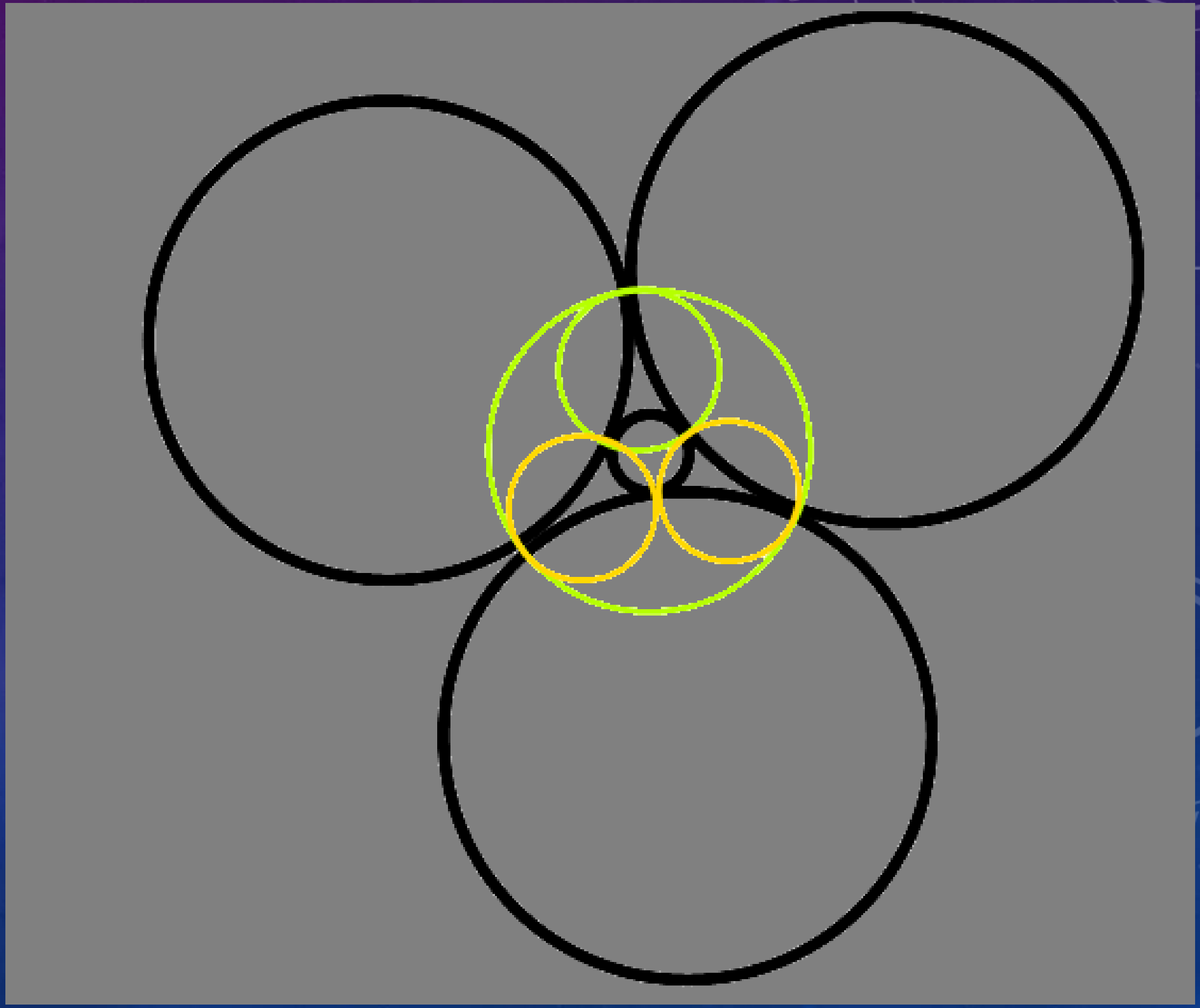
$$2 \sum b_i^2 = \left(\sum b_i\right)^2$$



PROOF OF DESCARTES' THEOREM

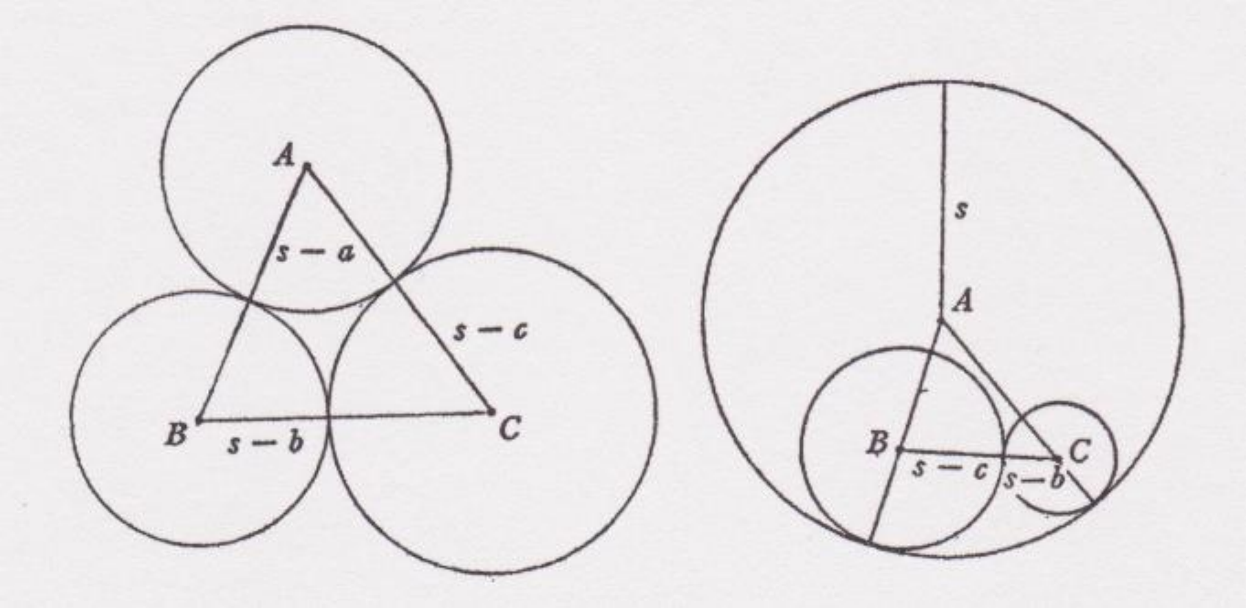
- To begin the proof we construct four additional circles, each one passing through three points of tangency of three original circles.
- Taking k_1, k_2, k_3, and k_4 to be the curvatures of the new circles, we apply the following:

$$2\left(k_1^2 + k_2^2 + k_3^2 + k_4^2\right) = \left(k_1 + k_2 + k_3 + k_4\right)^2$$



PROOF CONTINUED

- Let ABC be a triangle with sides a, b, and c. Let s be its semiperimeter and let r be the radius of the incircle (inradius) such that $r^2 = (s-a)(s-b)(s-c) / s$
- Any three mutually tangent circles can be considered as having centers A, B, and C with radii s-a, s-b, and s-c respectively.



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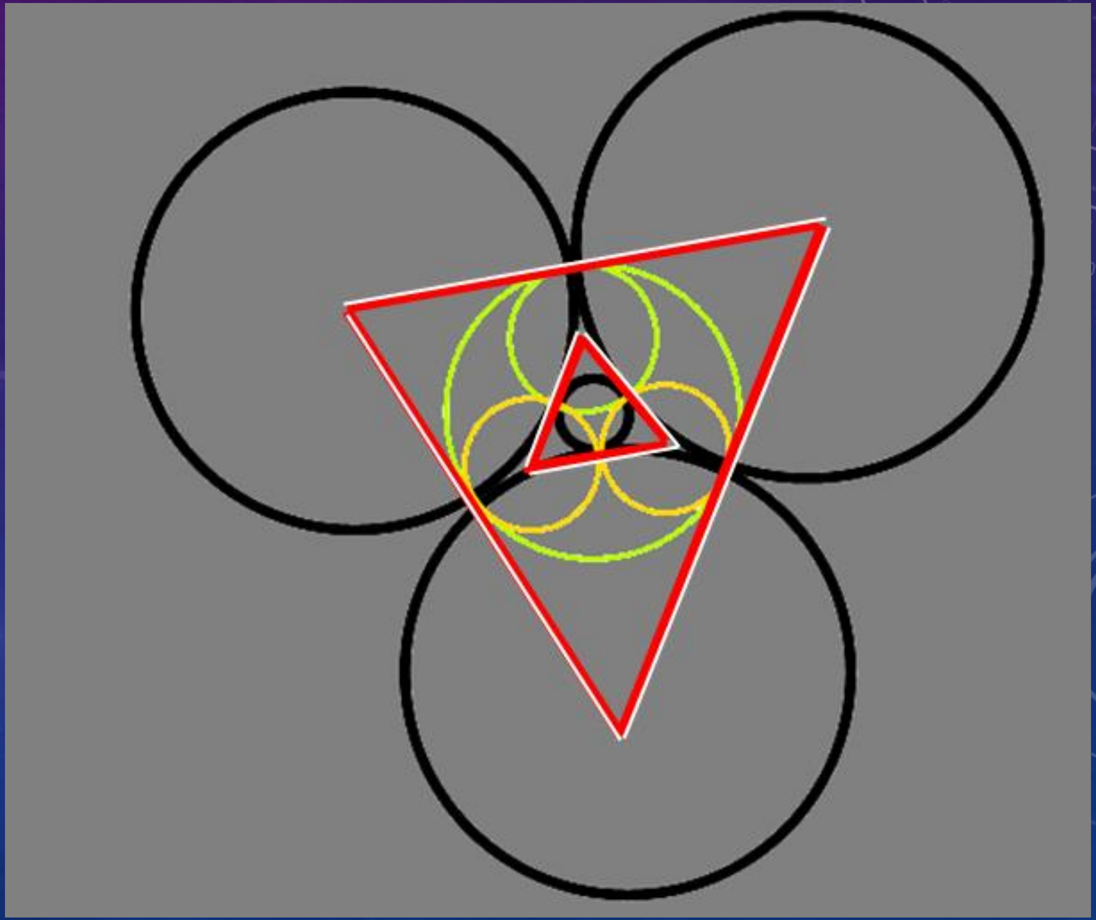
EXTERNAL CONTACT

- In this case let $1/k_1 = r$, $1/b_2 = s-a$, $1/b_3 = s-b$, and $1/b_4 = s-c$
- This is done in order to form a relationship between the first set of 4 mutually tangent circles and the second set.

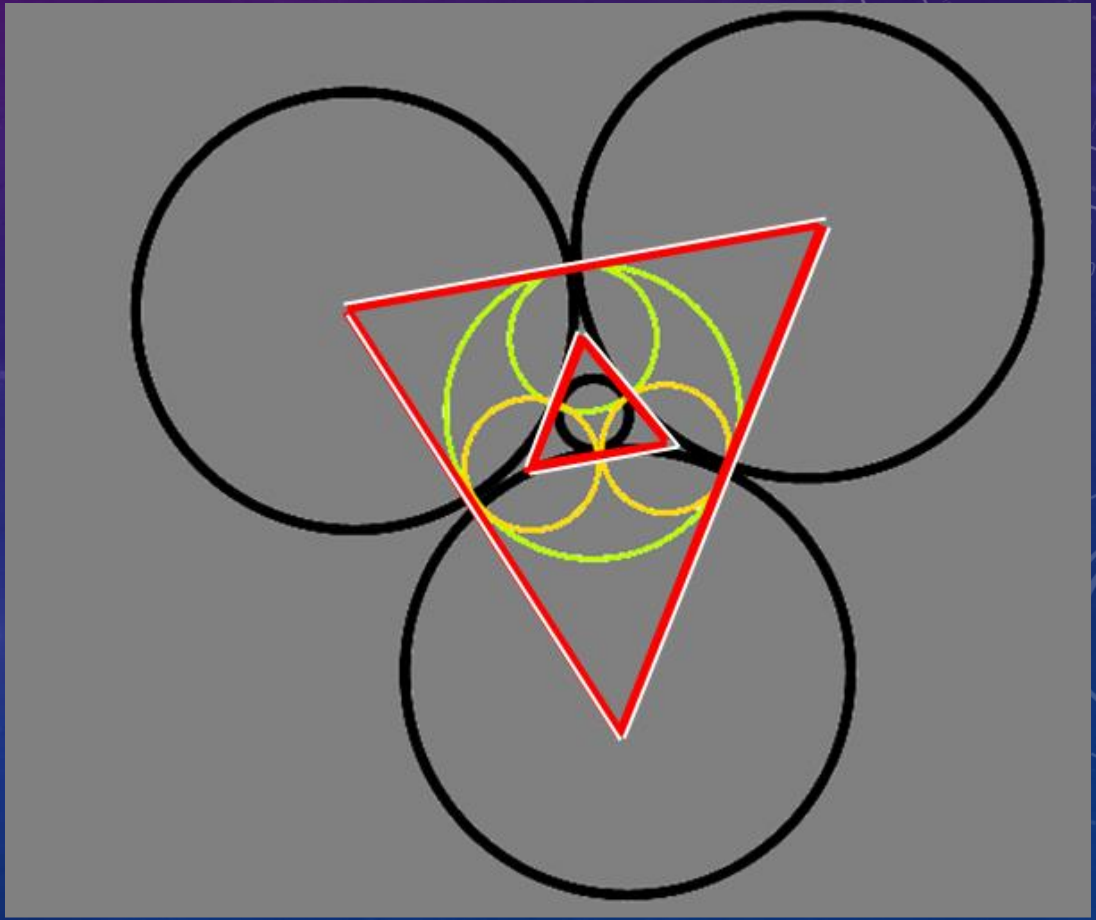


EXTERNAL CONTACT CONTINUED

- Expanding the right hand side of the first equation of curvatures, we see the following:

$$(\sum b_i)^2 = \sum b_i^2 + 2\sum b_i b_j$$

Proof continued on the board and in our paper…



CONNECTION TO THE APOLLONIAN GASKET

- An immediate consequence of the Descartes' circle theorem is that given the curvature of three mutually tangent circles, we can solve for the curvatures of the two circles that are mutually tangent to all three original circles. From this new collection of four mutually tangent circles, we can then arbitrarily choose three of them, and solve for the curvature of two new circles that are mutually tangent to this selection of three circles.
- This process can be repeated indefinitely, which is the process used for constructing an Apollonian Gasket.