Pick's Theorem

Math 445 Spring 2013 Final Project

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Pick's Theorem provides a simple formula for the area of any lattice polygon. A lattice polygon is a simple polygon embedded on a grid, or lattice, whose vertices have integer coordinates, otherwise known as grid or lattice points. Given a lattice polygon *P*, the formula involves simply adding the number of lattice points on the boundary, *b*, dividing *b* by 2, and adding the number of lattice points in the interior of the polygon, *i*, and subtracting 1 from *i*. Then the area of P is $\frac{b}{2} + i - 1$.

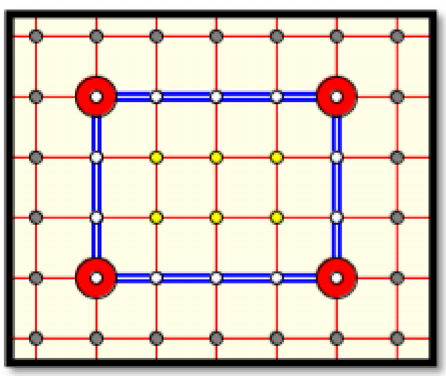
The theorem was first stated by Georg Alexander Pick, an Austrian mathematician, in 1899. However, it was not popularized until Polish mathematician Hugo Steinhaus published it in 1969, citing Pick. Georg Pick was born in Vienna in 1859 and attended the University of Vienna when he was just 16, publishing his first mathematical paper at only 17 (The History Behind Pick's Theorem). He later traveled to Prague where he became the Dean of Philosophy at the University of Prague. Pick was actually the driving force to the appointment of an up-and-coming mathematician, Albert Einstein, to a chair of mathematical physics at the university in 1911 (O'Connor). Pick himself ultimately published almost 70 papers covering a wide range of topics in math such as linear algebra, integral calculus, and, of course, geometry. His name still frequently comes up in studies of complex differential equations and differential geometry with terms like 'Pick matrices,' 'Pick-Nevanlinna interpolation,' and the 'Schwarz-Pick lemma.' He is, however, most remembered for Pick's Theorem, which he published in his 1899 paper, "*Geometrisches zur Zahlenlehre"* (The Geometric Theory of Numbers), in *Sitzungber. Lotos, Naturwissen Zeitschrift* (Sitzungber. Lotus, Natural Science Journal). Pick retired in 1927 and returned to Vienna, but fled to Prague in 1938 when the Nazis invaded Austria. Tragically, after the Nazis invaded Czechoslovakia in 1939, Pick was sent to Theresienstadt concentration camp in 1942 where he finally perished at 82 years old.

Steinhaus included Pick's Theorem in his famous book, *Kalejdoskop matematyczny* (Mathematical Snapshots), published in 1969, at which point the theorem garnered much more attention than it did during Pick's lifetime.

The Proof:

Pick's Theorem states: Let P be a lattice polygon, and let B(P) be the numbe of lattice points that lie on the edges of P and I(P) be the number of lattice points that lie on the interior of P. Then the area of P, denoted A(P) is equal to $\frac{B(P)}{2} + I(P) - 1$. This theorem allows one to find the area of any lattice polygon, or a polygon whose vertices lie on points whose coordinates are integers, known as lattice points, with one simple equation.

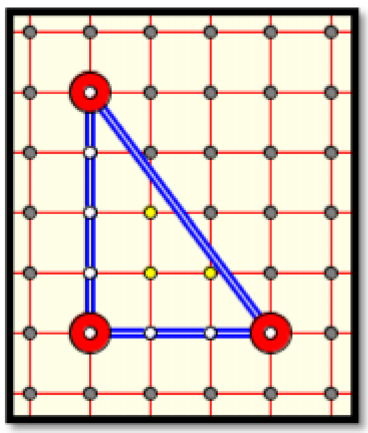
This implies that the area of lattice polygons is always half-integers. We will prove the theorem for rectangles, triangles, and then polygons of n-sides. The theorem is quite easy to prove for lattice rectangles.



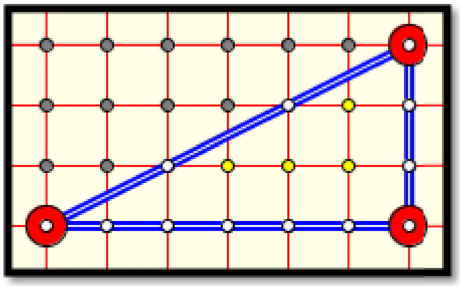
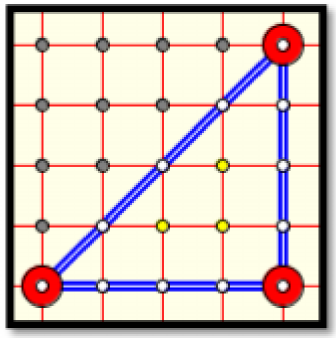
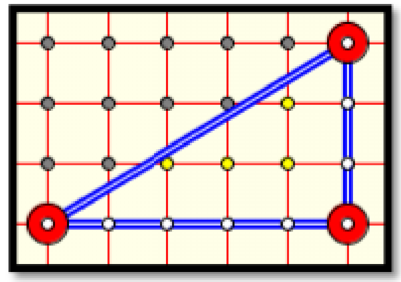
Part :1 Rectangles

In this section we will discuss only rectangles whose sides coincide with lattice lines. Other rectangles will be proven separately. Let P be a 3x4 lattice rectangle. Lemma 11.5 tells us that the area of this rectangle is (base)x(height) 3x4 = 12. If we apply Pick's Theorem, we have B(P) = 14 and I(P) = 6, so $A(P) = \frac{B(P)}{2} + I(P) - 1 = \frac{14}{2} + 6 - 1 = 12$. Now suppose P is a general $m \times n$ rectangle, where m, n are the numbers of lattice points contained in the base and height, respectively. In terms of units, where a unit is the distance between any two consecutive lattice points along a lattice line, P has $(m-1)(n-1)$ interior lattice points and 2m + 2n boundary lattice points. Therefore $A(P) = \frac{B(P)}{2} + I(P) - 1 = +(m-1)(n-1) - 1 = m + n + mn - m - n + 1 - 1 = mn.$

Part 2: Right Triangles



We will now prove that Pick's Theorem can be applied to right triangles, where again the triangles' legs are parallel the grid lines. If we view our right triangle as half of the $m \times n$ rectangle that we just proved with the addition of a diagonal, it becomes easier to prove. Lemma 11.9 tells us that the area of a right triangle is the product of its legs divided by 2, or half the area of the $m \times n$ rectangle. Therefore if we form a right triangle by connecting to of the opposite verticies of our $m \times n$ rectangle, the area is $\frac{mn}{2}$. Let T be this lattice right triangle. It is easy to count the number of lattice points along both of the triangles legs, but counting the lattice points along the hypotenuese can be trickier. As shown below, the hypotenuse can go through some, many, or even no lattice points. However, we will see that the number of points on the diagonal is not important.

Let k be the the number of lattice points the hypotenusecontains, excluding the two points at the verticies. Then the number of boundary points is m + n + 1 (the vertex at the right angle) + k. As we saw in Part 1, the $m \times n$ rectangle has $(m-1)(n-1)$ inteior points, so if we subtract from this the k points on the hypotenuse, the additional interior points are split between the two right triangles we formed. Therefore the right triangle we are interested in has

$\frac{(m-1)(n-1)-k}{2}$ interior points.

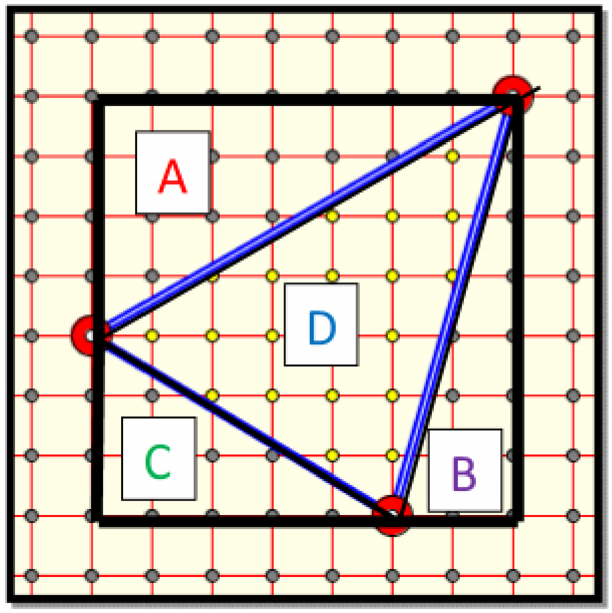
Checking Pick's Theorem for lattice right triangles we see that $A(T) = \frac{B(T)}{2} + I(T) - 1 =$

$$\frac{m+n+1+k}{2} + \frac{(m-1)(n-1)-k}{2} - 1 = \frac{m}{2}+\frac{n}{2}+\frac{1}{2}+\frac{k}{2}+\frac{mn}{2}-\frac{n}{2}-\frac{m}{2}+\frac{1}{2}-\frac{k}{2} - 1 = 1 + \frac{mn}{2} - 1 = \frac{mn}{2}.$$

This is what we expected to obtain, therefore Pick's Theorem can be applied to lattice right triangles.

Part 3: General Triangles

We will now show that Pick's Theorem can be applied to general triangles.



Knowing that Pick's Theorem works for rectangles and right triangles, we can prove that it works for arbitary triangles. When we proved that Pick's Theorem applies to the area of right triangles, we assumed that the triangle had two sides running directly through the two lattice lines in order to prove that we could create a lattice polygon. We wish to continue to use this idea by enclosing generic triangles in rectangles, so we now need to prove that Pick's Theorem applies to triangles that have just one side parallel to a lattice line and those with no sides parallel to lattice lines. . In reality there are numerous cases to consider, but any triangle without anyside lying directly on either of the lattice verticieswill all look more or less like our variation above, where an abitary triangle, D, can be extended into a rectangle with the addition of a few right triangles. In this case three right triangles are required: A, B, and C.

Let triangle A have interior points I_A and boundary points B_A, triangle B have interior points I_B and boundary points B_B, ect. Let the rectangle R have interior points I_R and boundary points B_R. We want to prove that $A(D) = B_D/2 + I_D - 1$.

Since we know Pick's Theorem applies to right triangles and rectangles, we know:

$A(A) = B_A/2 + I_A - 1$

$A(B) = B_B/2 + I_B - 1$

$A(C) = B_C/2 + I_C - 1$

$A(R) = B_R/2 + I_R - 1$

Lemma 11.2 tells us that the area of R is the sum of the area of its decomposition. Therefore,

$A(R) = A(A) + A(B) + A(C) + A(D)$

If we solve for the area of D we find:

$A(D) = A(R) - A(A) - (B) - A(C)$

(1) $A(D) = I_R - I_A - I_B - I_C + (B_R - B_A - B_B - B_C)/2 + 2$

Suppose our rectangle R is a m x n rectangle. Therefore it has area $A(R) = mn$, with $B_R = 2m + 2n$, and $I_R = (m-1)*(n-1)$. Because our rectangle R has common sides with triangles A, B, and C, and all three sides of our triangle D share sides these three triangles. We see that

(2) $B_R + B_D = B_A + B_B + B_C$ or

(3) $B_R = B_A + B_B + B_C - B_D$

Counting the interior points of our rectangle R we find:

(4) $I_R = I_A + I_B + I_C + I_D + (B_A + B_B + B_C - B_R) - 3$

The subtraction of 3 accounts for the fact that the vertices of triangle D are double counted. Substituting the value of B_R from equation (3) into equation (4) leads to the result:

(5) $I_R = I_A + I_B + I_C + I_D + B_D - 3$

Now if we substitude the value of I_R and B_R from equations (5) and (3), into equation (1), with a bit of alegbra we find:

$A(D) = I_R - I_A - I_B - I_C + (B_R - B_A - B_B - B_C)/2 + 2$

$A(D) = I_A + I_B + I_C + I_D + B_D - 3 - I_A - I_B - I_C + (B_R - B_A - B_B - B_C)/2 + 2$

$A(D) = I_D + B_D - 3 + (B_R - B_A - B_B - B_C)/2 + 2$

$A(D) = I_D + B_D - 3 + (B_A + B_B + B_C - B_D - B_A - B_B - B_C)/2 + 2$

$A(D) = I_D + B_D - 3 + (-B_D)/2 + 2$

$A(D) = I_D + (B_D - [B_D/2]) - 3 + 2$

$A(D) = I_D + B_D/2 - 1$

This final result is exactly what we wished to prove.

Now suppose one side of our triange lies on either of the lattice verticies and the other two do not.

We can form a rectangle by drawing two right triangles whose hypotenuses are the remaining two sides of the triangle, since any triangle has two accute angles. The proof that Pick's Theorem applies to such a triangle follows directly from the previous proof except we can eliminate triangle C, meaningwe only have to subtract 2 in equation (4), since only two vertices will be counted twice.

Part 4: Overview

We have already showed that Pick's Theorem is true for lattice rectangles and three sided lattice polygons. We now need to prove that Pick's Theorem is true for 4, 5, 6 … k – 1 sided polygons and therefore it is also valid for k-sided polygons. Because this will amount to infinite work, we will simply prove that Pick's Theorem is additive. That is, if we can apply Pick's Theorem to two polygons, then we can also apply it if we connect the two polygons.

Suppose we have a polygon P that can be subdivided into two polygons P_1 and P_2. Let P_1 have I_1 interior points and B_1 boundary points,and P_2 have I_2 interior points and B_2 boundary points. Assume that the common diagonal of P_1 and P_2 has d lattice points. Let P have total B boundary points and I interior points. Thus Pick's Theorem states:

$A(P) = A(P_1) + A(P_2) = (I_1 + B_1/2 - 1) + (I_2 + B_2/2 - 1)$

Clearly any interior point of P_1 or P_2 is also an interior point of P, and since two of the diagonal points lie on the boundary of P, d – 2 is the number of common boundary points of P_1 and P_2 are also interior points to P. Therefore, $I = I_1 + I_2 + d - 2$. Similar logic show that $B = B_1 + B_2 - 2(d - 2) - 2$.

Therefore:

$$I + \frac{B}{2} - 1 = I_1 + I_2 + d - 2 + \frac{B_1 + B_2 - 2(d-2) - 2}{2} - 1$$

$$= I_1 + I_2 + d - 2 + \frac{B_1}{2} + \frac{B_2}{2} - d + 2 - 1 - 1$$

$$= I_1 + I_2 + \frac{B_1}{2} + \frac{B_2}{2} - 2$$

$$= \left(I_1 + \frac{B_1}{2} - 1\right) + \left(I_2 + \frac{B_2}{2} - 1\right)$$

$$= A(P_1) + A(P_2) = A(P)$$

Part 5: Inner Diagonal

Now we wish to prove that Pick's Theorem applies to convex polygons. Theorem 10.19 tells us that in a convex polygon with n sides, the measure of the internal angles is 180(n-2). Therefore we can bisect any convex polygon into (n – 2) non-overlapping triangles, because the sides are semi parallel and a triangle has internal measure of 180. Since Pick's Theorem works for general triangles and it has an additive property, Pick's Theorem also applies to general convex polygons.

We can also generalize Pick's Theorem to certain non-convex polygons. If, for any polygon, every angle ABC, the angle with measure less than 180 is within the polygon, then we can bisect the polygon into triangles and Pick's Theorem applies. The proof has two cases:

1) Every side of the triangle is within the polygon, there for we can split out polygon into n-2 triangles and Pick's Theorem applies.
2) There exists a side of the polygon AC that does not live within the polygon, therefore we need to prove that Pick's Theorem applies to polygons with holes in them.

Part 6: Polygons with Holes

First, we will assume that we have a simple polygon P that has interior points I and boundary points B, therefore $A(P) = I + \frac{B}{2} - 1$. Let's now insert a 'hole' of dimensions 1x1, or a 1x1 polygon made up of interior points of P, into the polygon. When inserting a 1x1 rectangle into our polygon we would expect the area to decrease by 1 area unit. A 1x1 hole is a rectangle with area 1, it has B_h=4 and I_h=0. So A (H) = B_h/2+I_H -1= 4/2 +0-1= 1.

We would expect out polygon with the hole P_H, to have area $A(P_H) = A(P) - A(H) = I + \frac{B}{2} - 1 - \left(\frac{B_H}{2} + I_H - 1\right) = I + \frac{B}{2} - 1 - \left(\frac{4}{2} + 0 - 1\right) = I + \frac{B}{2} - 1 - 1 = I + \frac{B}{2}$, exactly what we wanted to prove

Therefore, Pick's Theorem applies to polygons, and polygons with a single hole in them. We can generalize our theorem to include polygons with any number of holes in them, or a hole of any dimensions. By viewing the hole as simply another polygon, we can find the area of the original polygon sans the hole and subtract the area of the hole, both of which we can find using Pick's Theorem. Thus we have the area of the polygon with a hole.

An Application of Pick's Formula

It can be proven by the use of Pick's Formula that the minimum possible area of any convex lattice pentagon ABCDE is 5/2.

Consider the vertex A. Since every integer is either odd or even there are exactly four possibilities: either both the x and y coordinates of A are even, both are odd, the x-coordinate is even and the y-coordinate is odd, or the x-coordinate is odd and the y-coordinate is even. In summary, A belongs to one of the following four classes: (Odd, Odd), (Even, Even), (Odd, Even) and (Even, Odd). Clearly, this is also true of the other four vertices (Kedlaya, Poonen, and Vakil 118-120).

Recall that the sum of two odd or two even numbers is even while the sum of an even and an odd is odd. If we call this property of an integer its "parity" we can summarize with the statement that, if two integers have the same parity, their sum is even and if two integers have different parities their sum is odd.

Consider the midpoint formula which states that in the Cartesian Model if M is the midpoint of $\overline{QR}$ and where $Q = (x_Q, y_Q)$ and $R = (x_R, y_R)$ then $M = \left(\frac{x_Q + x_R}{2}, \frac{y_Q + y_R}{2}\right)$. If both sums $x_Q + x_R$ and $y_Q + y_R$ are even then M has integer coordinates. Hence, if two integers have the same parity, there sum is even. Hence the midpoint of two distinct points in the same parity class will have integer coordinates.

Notice that there are five vertices of ABCDE while there are only four "parity classes". By the pigeonhole principal then it can be concluded that at least two of the vertices of ABCDE belong to the same parity class. Without loss of generality, we can say that one of these two vertices is A. There are two cases: the point with the same parity as A is one of the adjacent vertices (B or E) or one of the other vertices (C or D). We can call these cases 1 and 2.

In case 1, we can state without loss of generality that A and B belong to the same parity class. Hence it can be concluded that the midpoint (M) of $\overline{AB}$ has integer coordinates. There are then two subcases: either M is in the same parity class as A and B, or it is not. We can call these cases 1a and 1b.

In case 1a, since M has the same parity of A and B, we know that the midpoints of $\overline{AM}$ and $\overline{MB}$, which we can call N and O, also have integer coordinates. Additionally, we know by the definition of a midpoint that N ≠ O because A*M*B and A*N*M and M*O*B. We also know by the definition of a polygon that M, N and O are distinct from C, D, and E. Hence, in Case 1a, there are at least additional three lattice points on the boundary. Hence I ≥ 0 and B ≥ 8. We can use Pick's formula to conclude: $Area \geq 0 + \frac{8}{2} - 1$. In case 1a $Area \geq 3$.

In case 1b, M belongs to a different parity class than A and B. By the pigeonhole principle, M and one of the remaining vertices (C, D, E) have the same parity class. We can rename this vertex F.

The midpoint of $\overline{AF}$ (N) has integer coordinates, since they belong to the same parity class. By the Polygon Splitting Theorem, we know that, since ABCDE is convex, the diagonal $\overline{AF}$ is in the interior of ABCDE. Hence N is a lattice point in the interior of ABCDE. Hence ABCDE has at least one lattice point in its interior (N) and at least one additional boundary point (M). Hence l ≥ 1 and B ≥ 6. We can use Pick's formula to conclude that $Area \geq 1 + \frac{6}{2} - 1$. In case 1b $Area \geq 3$.

In case 2, C or D belongs to the same parity class as A. Without loss of generality, we can say A and C have the same parity. Hence the midpoint (M) of the diagonal $\overline{AC}$ has integer coordinates. As in case 1b, we can conclude that this point is in the interior of ABCDE by the Polygon Splitting Theorem. Hence, in case , there is at least one interior lattice point for ABCDE. Hence l ≥ 1 and B ≥ 5. We can use Pick's formula to conclude $Area \geq 1 + \frac{5}{2} - 1$. In case 1b $Area \geq 5/2$.

We can summarize with the following chart:

Case	Min B	Min I	Min Area
1a	8	0	3
1b	6	1	3
2	5	1	5/2

Hence we can conclude that the area of any convex lattice pentagon is larger than or equal to 5/2.

Now all that remains is to show there is a lattice pentagon with area 5/2.Consider a pentagon with the vertices : (0,0) , (0,1), (1,2), (2,1) and (1,0).

See diagram:

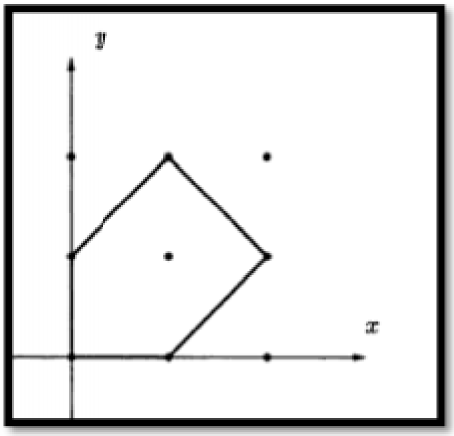


Image Source: (Kedlaya, Poonen, and Vakil 119)

As this pentagon is convex and has no additional boundary points and one interior point, we know that I = 1 and B = 5 hence $Area = 1 + \frac{5}{2} - 1 = \frac{5}{2}$. Hence it has been shown that there exists a convex pentagon with integer coordinates with area 5/2, and it has been proven that every convex pentagon with integer coordinates has an area of at least 5/2. Therefore, it can be concluded that the minimum area of a convex pentagon is 5/2.

Works Cited

A. Bogomolny, "Pick's Theorem", *Interactive Mathematics Miscellany and Puzzles* http://www.cut-the-knot.org/ctk/Pick_proof.shtml , Accessed 30 May 2013

"The History Behind Pick's Theorem." *School of Mathematics and Statistics University of St Andrews, Scotland*. University of Georgia. Web. 30 May 2013. <http://jsoles.myweb.uga.edu/history.html>.

Kedlaya, Kiran, Bjorn Poonen, and Ravi Vakil. *The William Lowell Putnam Mathematical Competition 1985-2000: Problems, Solutions, and Commentary*. Chicago: Mathematical Association of America, 2002. 118-120. Web. <http://books.google.com/books?id=AA-lOA1nPDcC&lpg=PA118&ots=8nOsFVKr2Z&dq=prove that any convex pentagon whose vertices have integer coordinates must have area at least 5 2 .&pg=PA120>.

O'Connor, JJ, and EF Robertson. "Georg Alexander Pick."*School of Mathematics and Statistics University of St Andrews, Scotland*. N.p., n.d. Web. 30 May 2013. <http://www-history.mcs.st-and.ac.uk/Biographies/Pick.html>.

Schultz, Kyle. "An Investigation of Pick's Theorem."*Department of Mathematics Education*. University of Georgia. Web.30 May 2013. <http://jwilson.coe.uga.edu/emat6680fa05/schultz/6690/pick/pick_main.htm>.