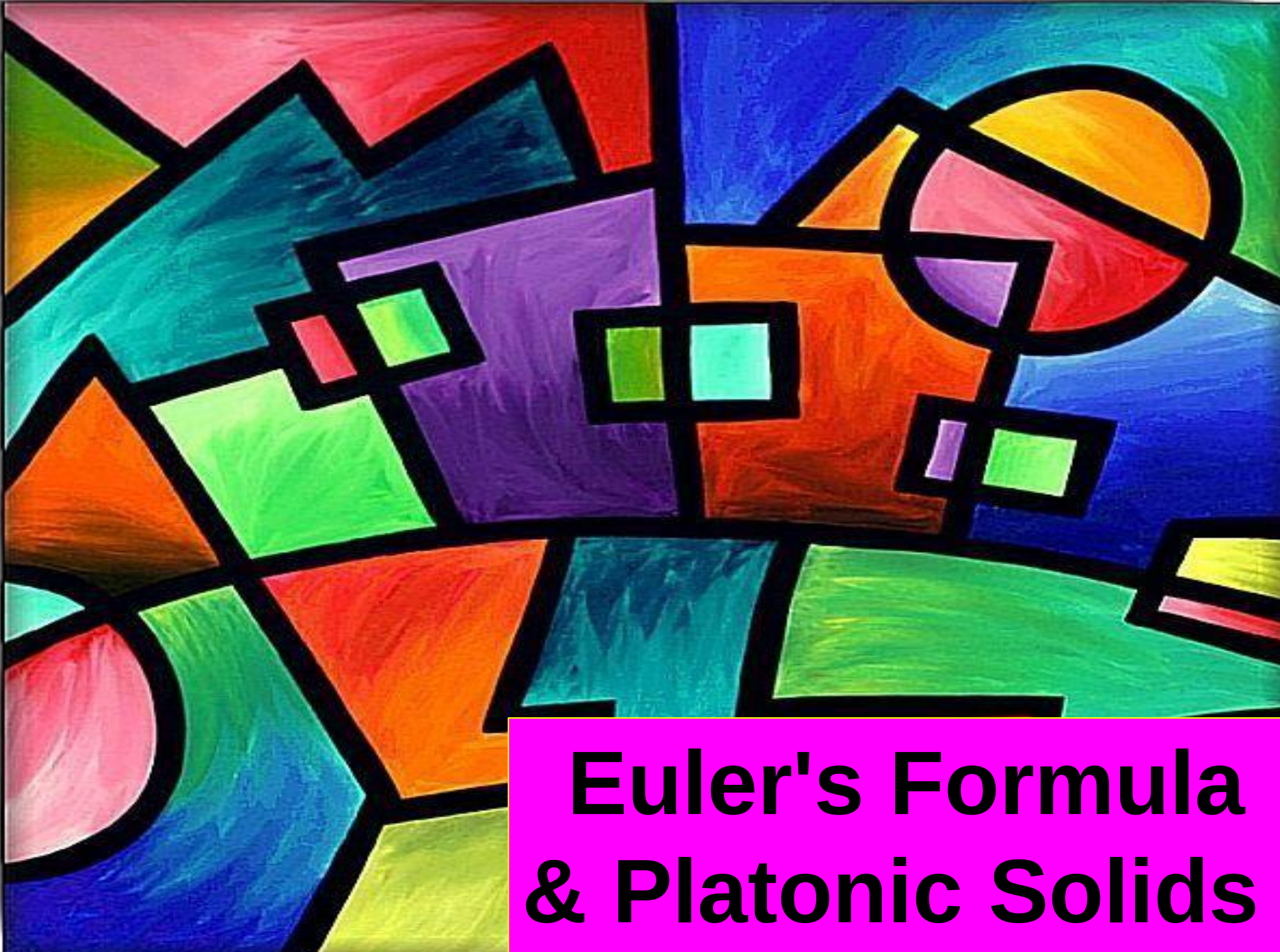
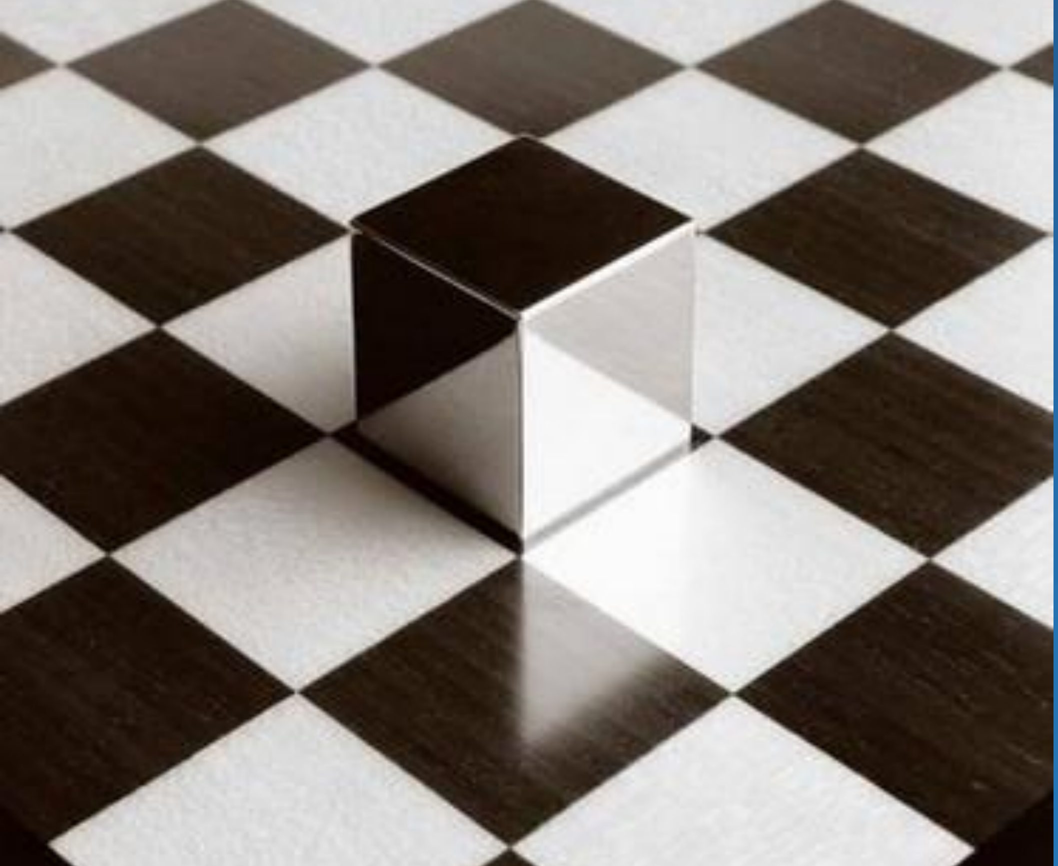


Euler's Formula & Platonic Solids

Introduction: Basic Terms



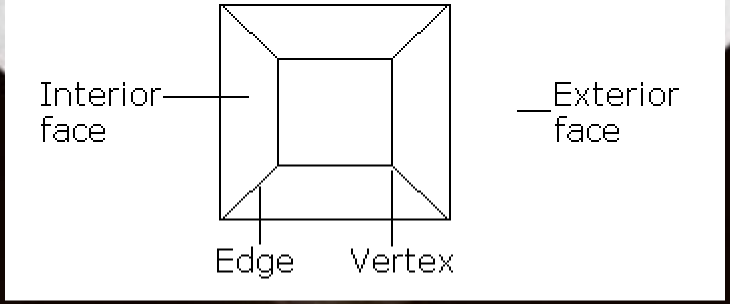
Vertices/Nodes: The common endpoint of two or more rays or line segments.

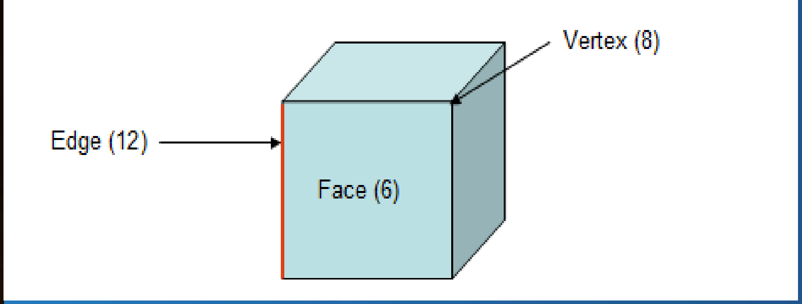
Edges: the line segments where two surfaces meet

Faces/Regions:

Interior: area containing all the edges adjacent to it

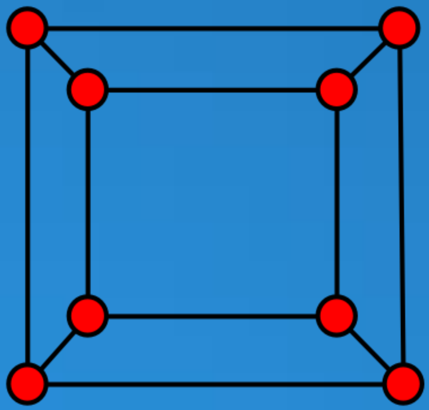
Exterior: the unbounded area outside the whole graph

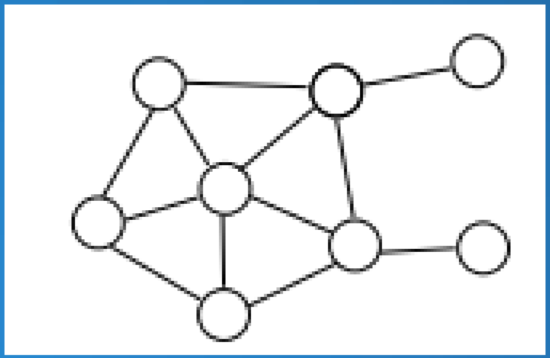




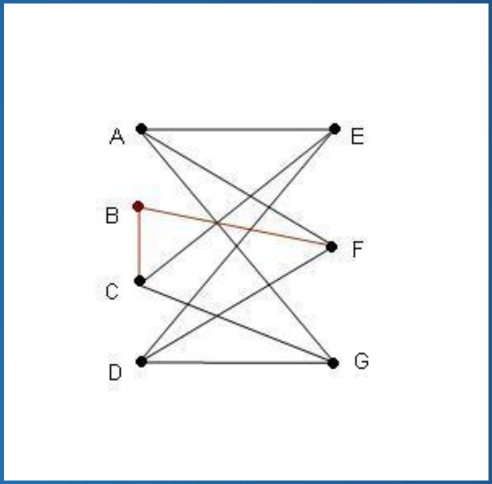
Definition:

A *planar graph* is one that can be drawn on a plane in such a way that there are no "edge crossings," i.e. edges intersect only at their common vertices.





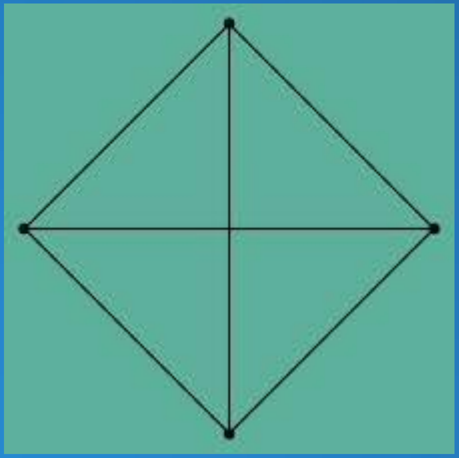
planar graph



non-planar graph

True or False:

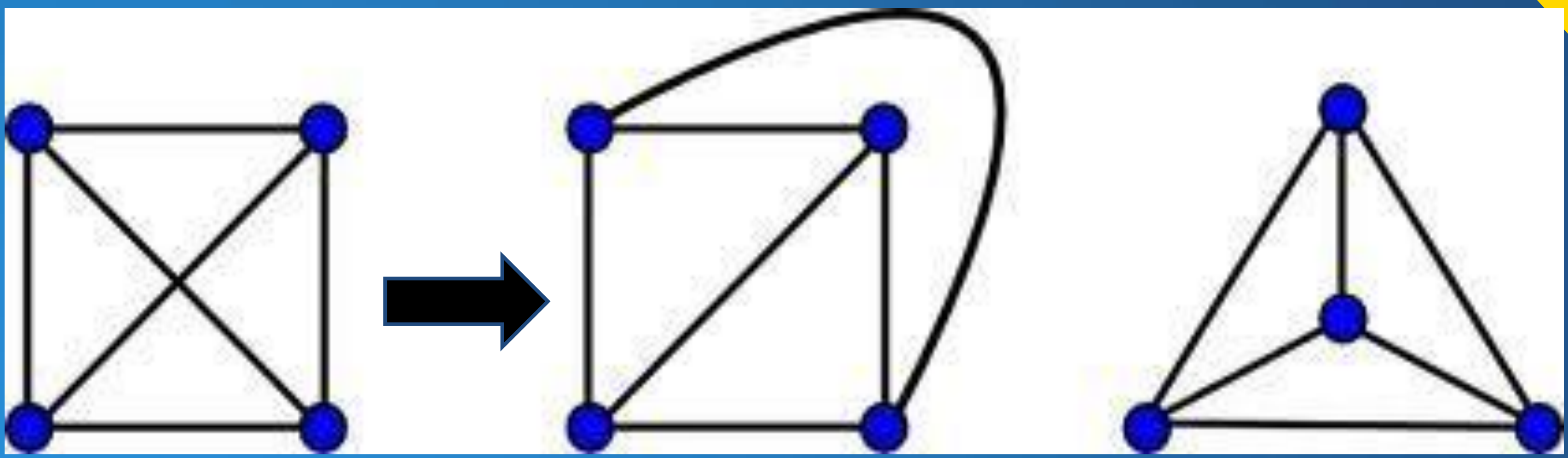
Is the following diagram a planar graph?



Answer: Yes.

Reason: Because a graph is also *planar* if the nodes of the graph can be rearranged (without breaking or adding any edges)

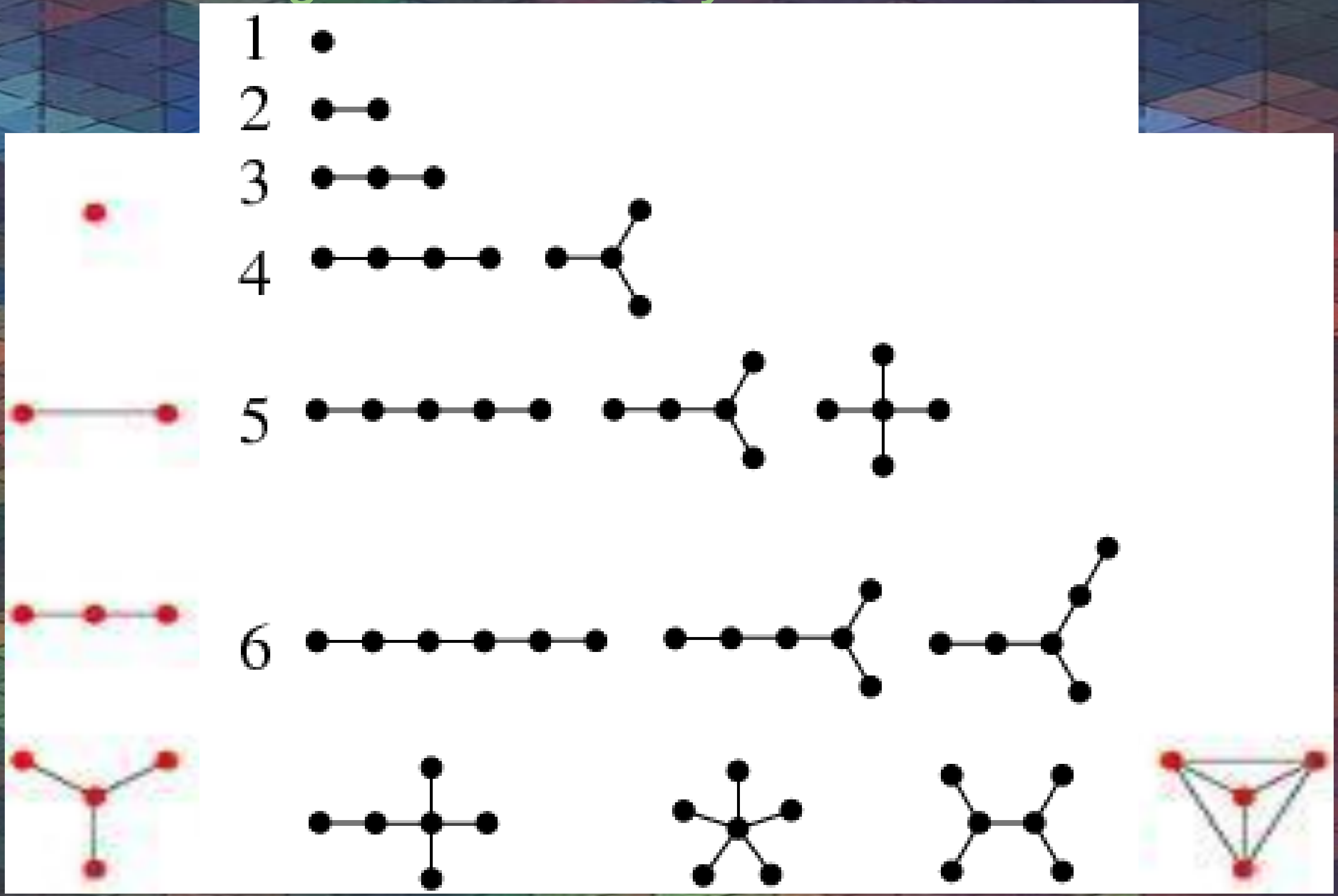
if we did the following changes:



http://www.flashandmath.com/mathlets/discrete/graphtheory/planargraphs.html

More Examples:

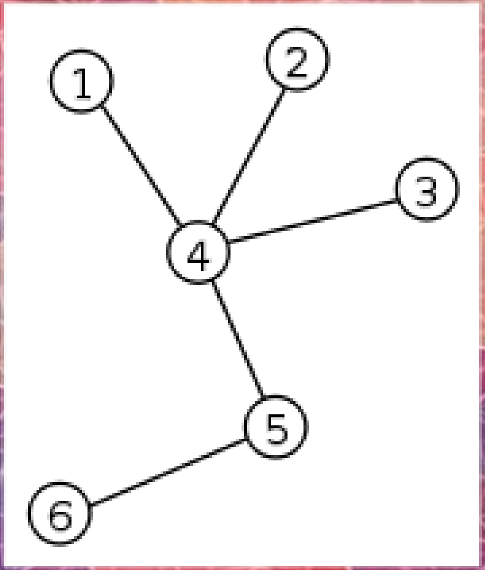
Increasing the number by vertices:



Special Planar Graph

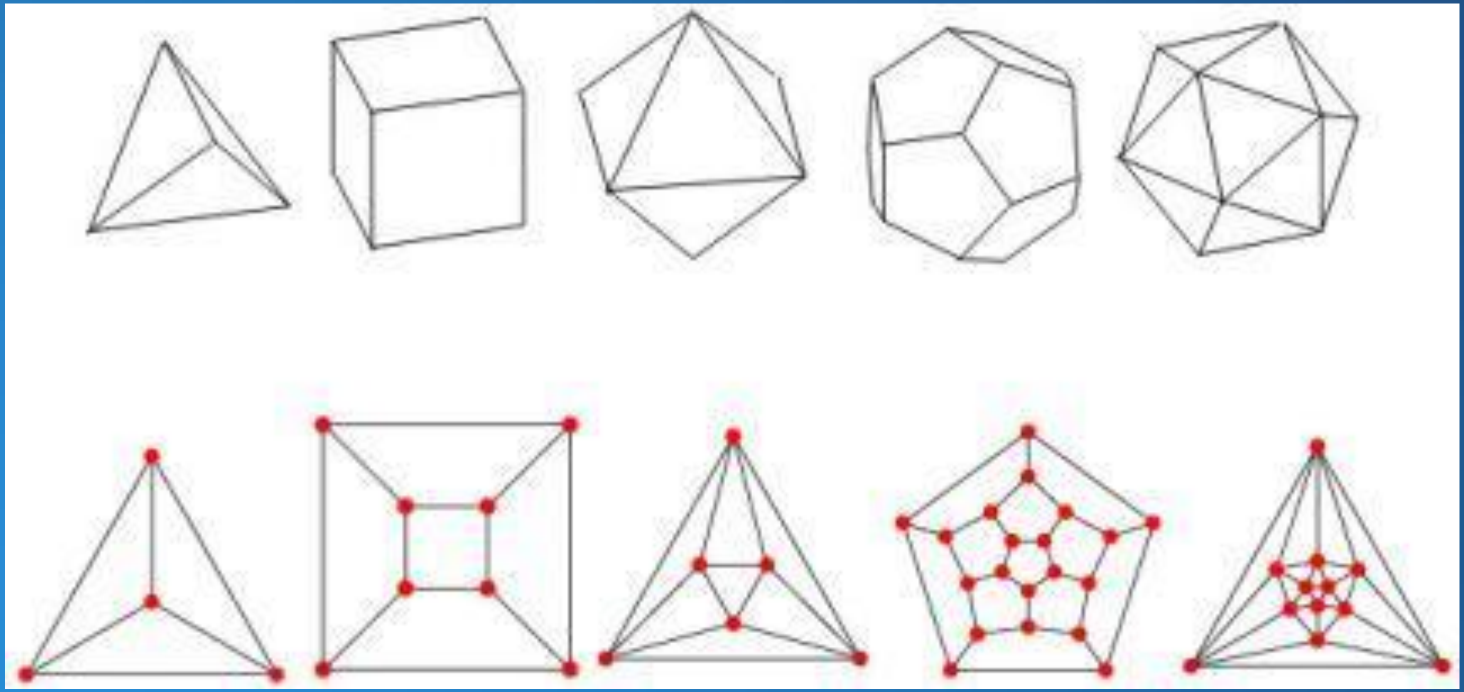
Tree: any connected graph with no cycles.

Notices: it only has an exterior face.



cycle: A *cycle* in a graph means there is a path from an object back to itself

Platonic Solids and Planar Graphs



Euler's Characteristic Formula
$V - E + F = 2$

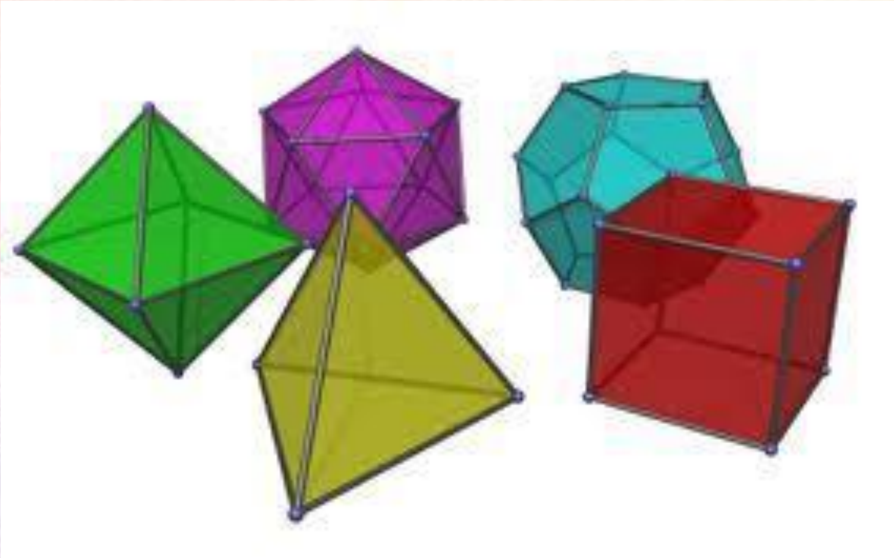
Euler's Characteristic Formula states that for any connected planar graph, the number of vertices (V) minus the number of edges (E) plus the number of faces (F) equals 2.

Platonic Solids

Definition

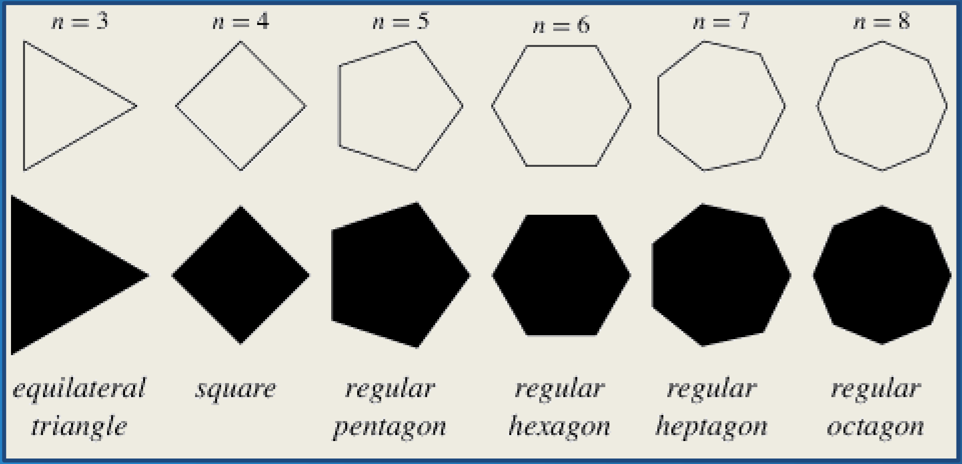
Platonic Solids:

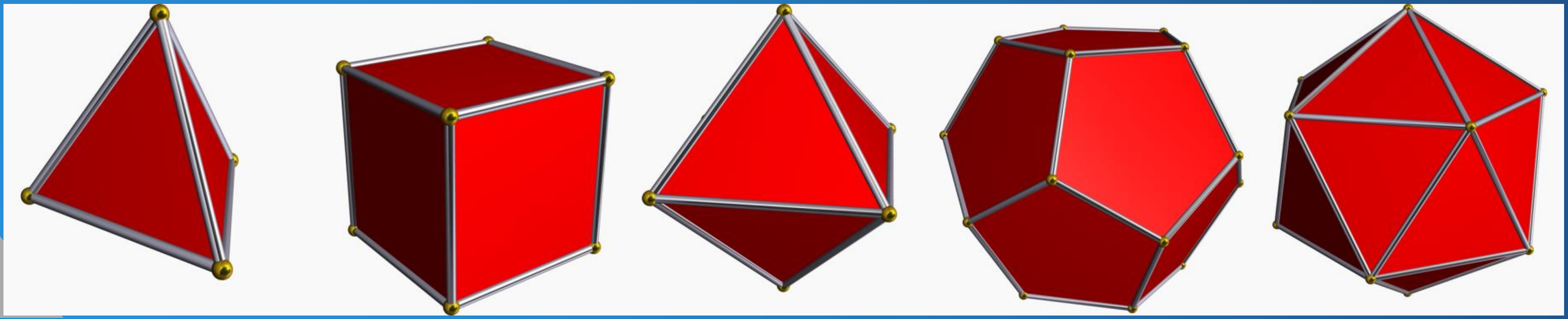
- Regular
- Convex
- Polyhedron

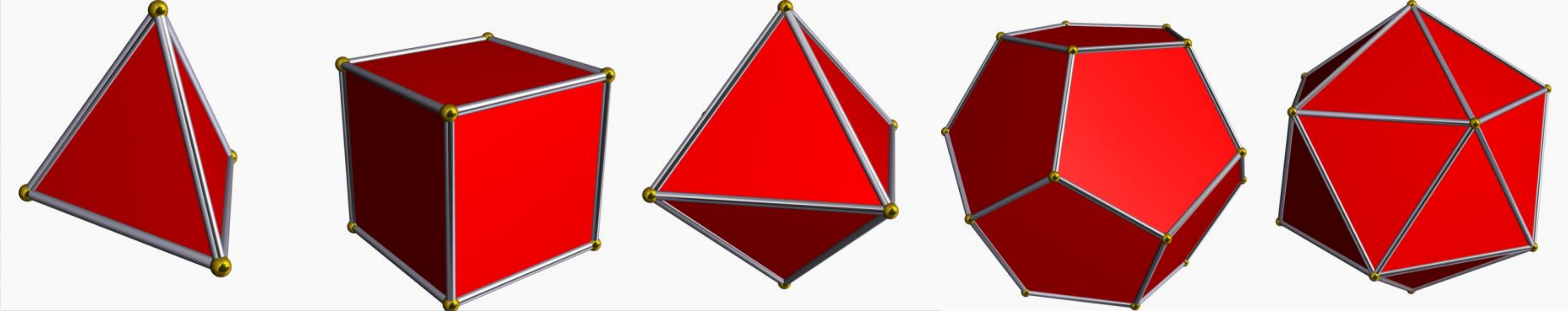


has regular polygon faces with the same number of faces meeting at each vertex.

How many regular convex polehedra are there?



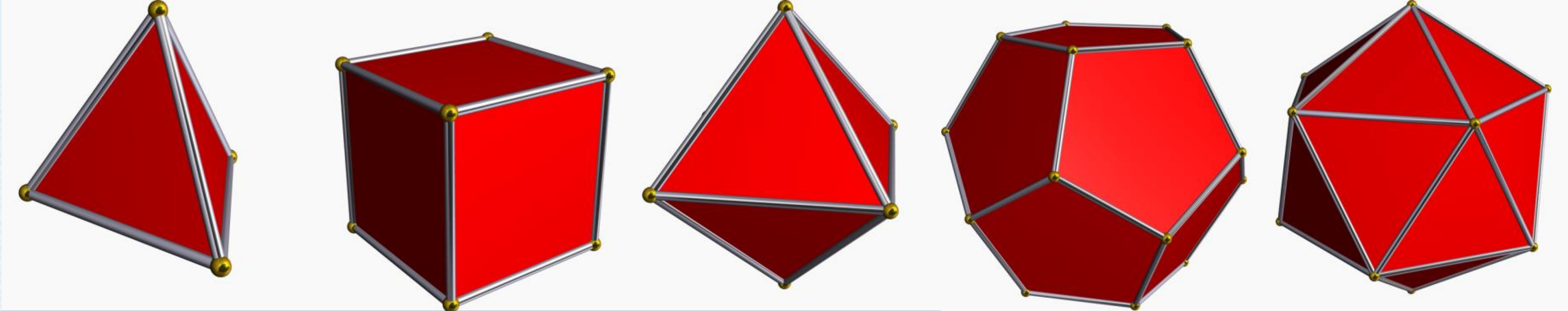




Name	Tetrahedron	Hexahedron (Cube)	Octahedron	Dodecahedron	Icosahedron
Vertices	4	8	6	20	12
Edges	6	12	12	30	30
Faces	4	6	8	12	20

The Five Platonic Solids

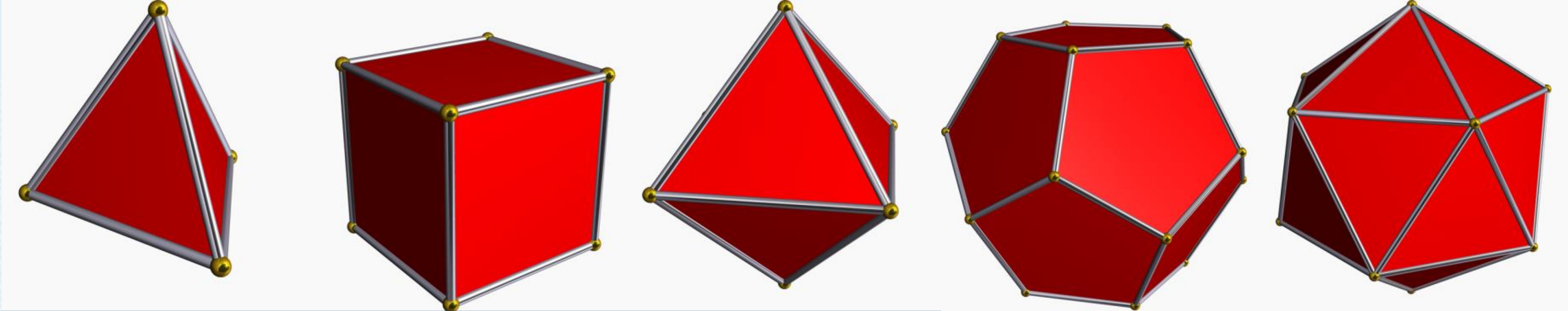
Tying it Together



Name	Tetrahedron	Hexahedron (Cube)	Octahedron	Dodecahedron	Icosahedron
Vertices	4	8	6	20	12
Edges	6	12	12	30	30
Faces	4	6	8	12	20

Euclidean Characteristic

$$V - E + F = 2$$



NameVertices	Tetrahedron	Hexahedron (Cube)	Octahedron	Dodecahedron	Icosahedron
Edges	4	8	6	20	12
Faces	6	12	12	30	30
V - E + F	4	6	8	12	20
	4 - 6 + 4 = 2	8 - 12 + 6 = 2	6 - 12 + 8 = 2	20 - 30 + 12 = 2	12 - 30 + 20 = 2

Euler's Formula Holds for all 5 Platonic Solids

Proof that there are only 5 platonic solids

Using Euler's Formula

Proof

- Let n be the number of edges surrounding each face
- Let F be the number of faces
- Let E be the number of edges on the whole solid

n: number of edges surrounding each face

F: number of faces

E: number of edges

Proof

So does F * n = E?

Not quite, since each edge will touch two faces, so F * n will double count all of the edges,

i.e. F * n = 2E

(F * n) / 2 = E

n: number of edges surrounding each face

F: number of faces

E: number of edges

Proof

(F * n) / 2 = E

So what is E in terms of the number of vertices?

- Let c be the number of edges coming together at each vertex
- Let V be the number of vertices in the whole solid

n: number of edges surrounding each face

F: number of faces

E: number of edges

c: number of edges coming to each vertex

V: number of vertices

Proof

(F * n) / 2 = E

So what is E in terms of the number of vertices?

So does E = V * c?

Not quite, since each edge comes to two vertices, so this will double count each edge

i.e. 2E = V * c

E = (V * c) / 2

n: number of edges surrounding each face

F: number of faces

E: number of edges

c: number of edges coming to each vertex

V: number of vertices

Proof

$(F * n) / 2 = E$

$(V * c) / 2 = E$

Euler's Formula:

$V - E + F = 2$

To use this, let's solve for V and F in our equations

n: number of edges surrounding each face

F: number of faces

E: number of edges

c: number of edges coming to each vertex

V: number of vertices

Proof

(F * n) / 2 = E

F = 2E / n

(V * c) / 2 = E

V = 2E / c

Euler's Formula:

V - E + F = 2

To use this, let's solve for V and F in our equations

$$\left(\frac{2E}{c}\right) - E + \left(\frac{2E}{n}\right) = 2$$

$$E\left(\frac{2}{c} - 1 + \frac{2}{n}\right) = 2$$

Part of being a platonic solid is that each face is a regular polygon. The least number of sides (n in our case) for a regular polygon is 3, so

$$n \geq 3$$

There also must be at least 3 faces at each vertex, so

$$c \geq 3$$

n: number of edges surrounding each face

F: number of faces

E: number of edges

c: number of edges coming to each vertex

V: number of vertices

Proof

F = 2E / n

V = 2E / c

$$E\left(\frac{2}{c} - 1 + \frac{2}{n}\right) = 2$$

Let's think about this equation

Since E is the number of edges, E must be positive, so

$$\left(\frac{2}{c} - 1 + \frac{2}{n}\right) > 0$$

n: number of edges surrounding each face $n \geq 3$

F: number of faces

E: number of edges

c: number of edges coming to each vertex $c \geq 3$

V: number of vertices

Proof

$$E\left(\frac{2}{c}-1+\frac{2}{n}\right)=2$$

$$\left(\frac{2}{c}-1+\frac{2}{n}\right)>0$$

Let's think about this equation

It will put some restrictions on c and n

$$\frac{1}{c}>\frac{1}{2}-\frac{1}{n}>\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

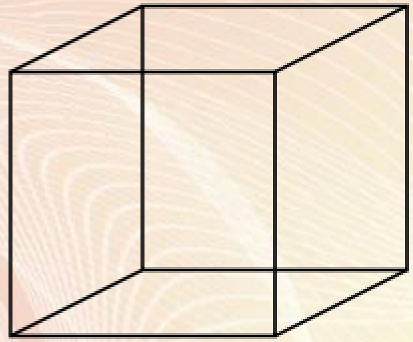
Since $n \geq 3$, we have that $\frac{1}{n} \leq \frac{1}{3}$

$$\frac{1}{c}>\frac{1}{6}$$

$$c<6 \Rightarrow 3 \leq c < 6$$

c = 3, 4, or 5

Now, watch carefully...



n: number of edges surrounding each face $n \geq 3$

F: number of faces

E: number of edges

s $c \geq 3$

coming to each vertex

V: number of vertices

Proof

$$E\left(\frac{2}{c}-1+\frac{2}{n}\right)=2$$

$$\left(\frac{2}{c}-1+\frac{2}{n}\right)>0$$

Let's think about this equation

It will put some restrictions on c and n

$$\frac{1}{n}>\frac{1}{2}-\frac{1}{c}>\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

Since $c \geq 3$, we have that $\frac{1}{c} \leq \frac{1}{3}$

$$\frac{1}{n}>\frac{1}{6}$$

$$n<6 \Rightarrow 3 \leq n < 6$$

n = 3, 4, or 5

n = 3, 4, or 5 $n \geq 3$

n: number of edges surrounding each face

F: n[illegible] of faces

E: number of edges

c = 3, 4, or 5

c: number of edges coming to each vertex

V: number of vertices

Proof

$$E\left(\frac{2}{c}-1+\frac{2}{n}\right)=2$$

$$\left(\frac{2}{c}-1+\frac{2}{n}\right)>0$$

When c = 3, n = 3, 4, or 5
When c = 4, n = 3
When c = 5, n = 3

$$\frac{1}{n}>\frac{1}{2}-\frac{1}{c}$$

When c = 3, $\frac{1}{n}>\frac{1}{6}$, so n < 6, so n = 3, 4, or 5

When c = 4, $\frac{1}{n}>\frac{1}{4}$, so n < 4, so n = 3

When c = 5, $\frac{1}{n}>\frac{3}{10}$, so n = 3

n = 3, 4, or 5

n: number of edges surrounding each face

F: number of faces

E: number of edges

c = 3, 4, or 5

c: number of edges coming to each vertex

V: number of vertices

Proof

$$E\left(\frac{2}{c} - 1 + \frac{2}{n}\right) = 2$$

When c = 3, n = 3, 4, or 5
When c = 4, n = 3
When c = 5, n = 3

c	*n*	*V*	*E*	*F*	**Solid**
3	3	4	6	4	Tetrahedron
3	4	8	12	6	Square
3	5	20	30	12	Dodecahedron
4	3	6	12	8	Octahedron
5	3	12	30	20	Icosahedron

The 5 Platonic Solids

n = 3, 4, or 5

n: number of edges surrounding each face

F: number of faces

E: number of edges

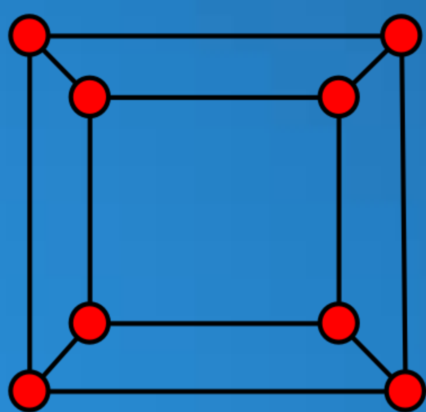
c = 3, 4, or 5

c: number of edges coming to each vertex

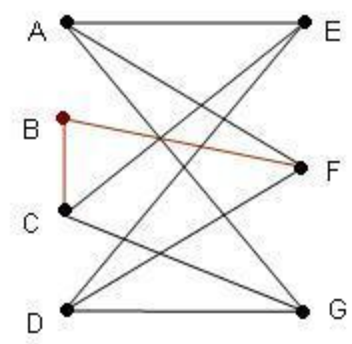
V: number of vertices

Remember this?

A *planar graph* is one that can be drawn on a plane in such a way that there are no "edge crossings," i.e. edges intersect only at their common vertices.



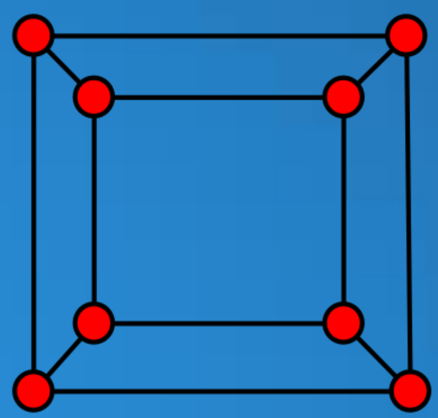
planar graph



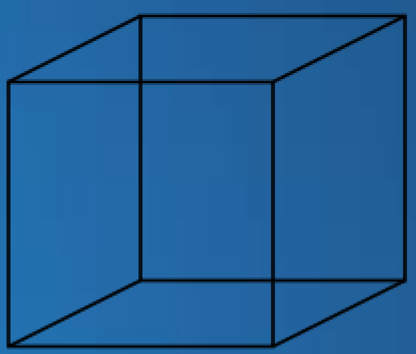
non-planar graph

Remember this?

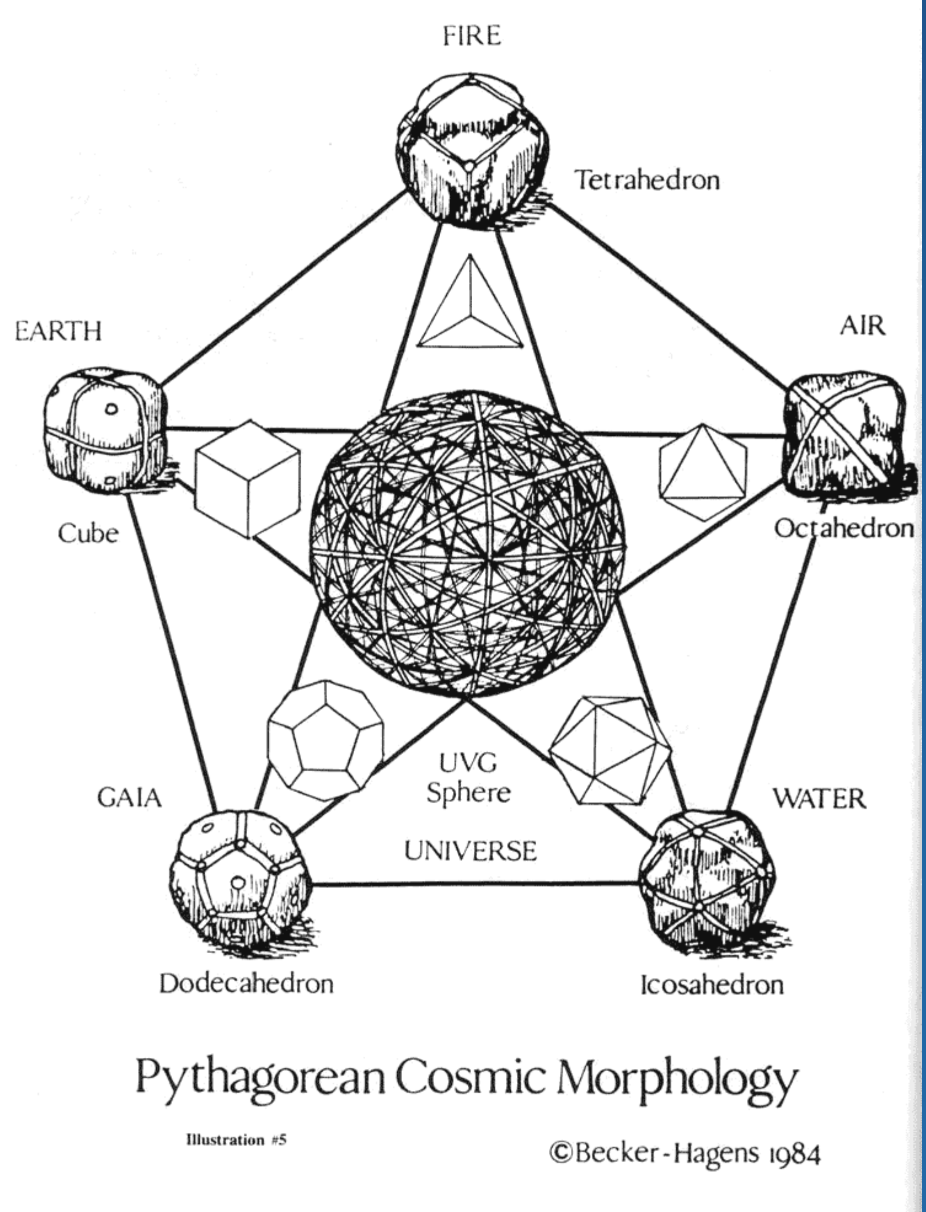
A *planar graph* is one that can be drawn on a plane in such a way that there are no "edge crossings," i.e. edges intersect only at their common vertices.



planar graph



Corresponding Platonic Solid


FIRE
Tetrahedron
EARTH
AIR
Cube
Octahedron
UVG
Sphere
GAIA
WATER
UNIVERSE
Dodecahedron
Icosahedron
Pythagorean Cosmic Morphology
Illustration #5
©Becker-Hagens 1984

Outline of visual to accompany proof by angle sums

1. Make planar graph using straight lines

2. Find total angle sum using polygon sums.

(n-2)180 *6F , n=4

Total sum = 360*6 = (2E-2F)180 = (2*12-2*6)180= 360*6

3. Find total angle sum using vertices

Interior vertices (4) = 360*4

Exterior vertices = 2(180-exterior angle)

Total sum = 360IV + 360EV -2*360

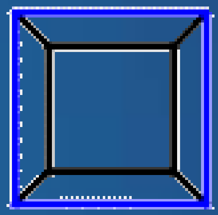
= 360V – 2*360 = 360*6

4. Set the equations equal to each other

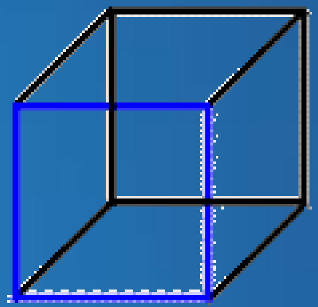
(2E-2F)180 = 360V – 2*360

Divide by 360 = E-F = V – 2

Rearrange V-E+F = 2



Shadow



Cube