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Math 445 A - Group Project

June 5th, 2013

Descartes and the Apollonian Gasket

Prompt

(1) Describe the construction of the Apollonian Gasket

(2) State and prove Descartes' theorem on four "kissing" circles. Explain the relevance to the Apollonian Gasket.

Part 1 – The Construction of the Apollonian Gasket

In order to describe the construction of the Apollonian gasket, we will first introduce some important terms:

- **Apollonian Gasket**: One of several names for a series of circles, nested inside one large circle, and tangent to all others nearby. Also called Soddy Circles or Kissing Circles.
- **Radius of a circle**: Distance from the center point to the edge of a circle.
- **Curvature of a circle**: Inverse of the radius; ±1/r, + when dealing with the outer curvature, or curvature along the outside, and - for the inner.
- **Tangent**: When two circles have a point in common. They must have the same slope at that point, so no intersection.
- **Descartes' Theorem**: The formula we'll be using to calculate all of our circles' sizes. It states: if the curvature of the first three circles are labeled c1, c2 and c3, respectively, the Theorem states that the curvature of the circle tangent to all three, d, is d = c1 + c2 + c3 ± 2 (sqrt (c1 * c2 + c2 * c3 + c3 * c1)).

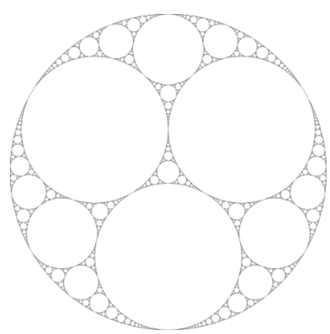


Figure 1 - The Apollonian Fractal

To begin, draw three small circles C1, C2, and C3, each one of which is tangent to the other two circles. We say that C1, C2, C3 are "mutually tangent." In the picture above, these would be the three largest internal circles. Then by Descartes' theorem, there exist two circles, C4 and C5, which are tangent to all three of C1, C2, and C3. Those two circles, C4 and C5, are called apollonian circles. In the picture, one of those, let's call it C4, is the outer circle, and the other, C5, is the small circle in the center of the original three. Since C4 is tangent to C1 and C2, we can apply the same process starting from those three circles, and the same goes for C5. In fact, we can do this with any set of three circles consisting of one circle from (C4, C5) and two from (C1, C2, C3). This gives us a pattern where beginning from 3 mutually tangent circles, we add 2 more (C4, C5) in one iteration (n=0) of this procedure. After a second iteration (n=1), we add (3 choose 2) * (2 choose 1) = 6. In general, we can add $2*3^n$ new circles at stage n, and giving a total of $3^n+1 +2$ circles after stage n. This infinite set of circles is the Apollonian gasket.

To calculate the curvature of the circle that is tangent to any set of 3 mutually tangent circles, we apply Descartes' Theorem as described. The curvature has three possible results: Negative curvature means that all other circles are *internally* tangent to that circle, like C4. Positive curvature means that all other circles which are tangent to that circle are *externally* tangent to that circle, like C5. Zero curvature means the "circle" is a line. In the case where we begin with 3 mutually tangent circles, this won't happen; however, it is valid to begin with one (or, in fact, two or even three; however, neither of these describe the classical Apollonian Gasket) of the mutually tangent "circles" being a line.

Finally, to compute the position of our new circle (C4 or C5, for example), we treat the centers as complex numbers of the form x + y*i* (*i* here is sqrt(-1)). We then let $c_i = p_i * 1/r_i$, and apply the complex version of Descartes' algorithm as usual. For a more obviously geometric method of finding the center of the fourth circle, consider that we have three mutually tangent circles, and the radius of a fourth. Now the distance from the center of a circle to another circle is the sum of their radii. At this step, we may recalculate the radii of circles by setting $r_i = 1/k_i$, so that the radius of a circle which has the other circles internal to it becomes negative, as its curvature is; this allows us to collapse that special case easily into the standard case for calculating distances. Now, choose one of the three initial circles, and call it C_1. Draw a circle around the center of C_1 with radius r_1 + r_4, where r_4 is the radius of our unknown-position fourth circle. Follow the same procedure for the other two circles. Now there will be up to two points where all three of the constructed circles intersect. Any points where this is the case can be the center of a circle with radius r_4, and be mutually tangent to the initial three circles!

There is another fractal called the Sierpinski triangle that is similar to the Apollonian gasket. The Sierpinski triangle is a fractal based on a triangle with four equal triangles inscribed in it. The central triangle is removed and each of the other three treated as the original was, and so on, creating an infinite regression in a finite space. The first step of creating a Sierpinski triangle is constructing a large equilateral triangle. The second step is finding the midpoint of each side of equilateral triangle, then connecting those midpoints to make a new triangle inside the bigger one. So far, we will have four equal triangles inside the bigger one, one of which will be inverted. Leave the upside down triangle alone, and apply the second step for other three triangles to make more small equilateral triangles. This is continued indefinitely to create a fractal pattern. Consider the state after the first iteration of this process, when there are three small equilateral triangles with one upside down in their center. This is quite similar to the situation when you have three mutually tangent circles in place of the upright triangles, and there is one circle in the center tangent to all three of them.

Figure 2 – The Sierpinski Triangle

Part 2 – The Descartes' Circle Theorem

When Rene Descartes first shared his circle theorem in 1643, his proof was incomplete. Many notable mathematicians had rediscovered it over the years independently and several different proofs have now been given. One such rediscovery was attributed to H. Beecroft, who published a complete proof in the journal, *Lady's and Gentlemen's Diary,* in 1842. In 1968, H. M. S. Coxeter published a simplification of Beecroft's proof in the *The American Mathematical Monthly*. For the most part, we will follow this proof by Coxeter in **[2]** for the following theorem. Before attempting the proof, we will need some definitions of some unfamiliar terms.

Definitions:

incircle – Given any triangle, the incircle is the largest circle contained in the triangle which is tangent to all three of the sides.

incenter – The center of the incircle.

inradius – the radius of the incircle.

excircle – Given a triangle, extend two sides in the direction opposite their common vertex. The circle tangent to both of these lines and to the third side of the triangle is called an excircle.

excenter – The center of the excircle.

exradius – The radius of an excircle.

semiperimeter – Half of the perimeter of a polygon (we will be using the semiperimeter of a triangle, hence $s=(a+b+c)/2$ for our purposes).

curvature (or "bends") of a circle – The curvature of a circle is defined as the reciprocal of the radius.

oriented circle – A circle with an assigned direction of unit normal vector, pointing either inward or outward. A positive radius implies an inward pointing normal vector (smaller circle), and a negative radius implies an outward pointing normal vector (larger circle).

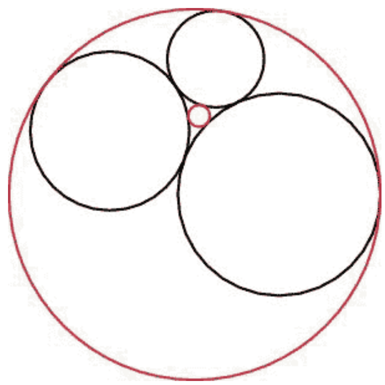


Figure 1. *Descartes configuration showing the two possible configurations: external contact and internal contact.*

Descartes' Circle Theorem. *In a Descartes configuration of four mutually tangent circles, the curvatures satisfy*

$$2 \sum b_i^2 = \left(\sum b_i\right)^2 \quad . \qquad (1.1)$$

It should be noted that the preceding theorem will apply to any Descartes configuration given that we define the curvature of the circles such that they are compatible in the sense that (i) the interiors of all four circles are disjoint, or (ii) the interiors are disjoint when all the *orientations* are reversed [**1**, pp. 339].

Proof:

The key to starting this proof was to recognize that the four mutually tangent circles are part of a configuration of eight circles, each passing through the three points of tangency of the three others.

[*Note: for all summation notation, let the indices i start at 1, n = 4, and i < j , for i, j = 1, 2, 3, 4. Equations labeled (1.xx*) refer you to this note.]

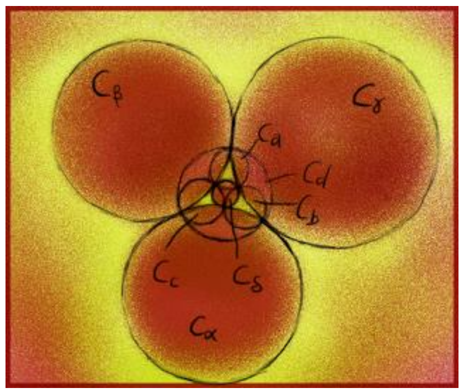
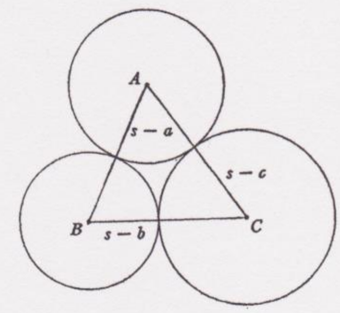


Figure 2. *configuration of eight circles*

Let b_1, b_2, b_3, b_4, k_1, k_2, k_3, and k_4 be the bends (curvatures) of these eight circles, such that the circles with bends b_i are mutually tangent, as are the circles with bends k_i, where i = 1, 2, 3, 4. By definition, the curvature is the reciprocal of the radius of a given circle. Then the theorem states $2\sum b_i^2 = (\sum b_i)^2$, holds. Similarly, $2 \sum k_i^2 = (\sum k_i)^2$, also holds. Let a, b, c, s, r and r_e denote the sides (a, b, c), semiperimeter (s), inradius (r), and first exradius (r_e) of a triangle ABC, so that $r^2=(s-a)(s-b)(s-c)/s$ and $r_e^2=s(s-b)(s-c)/(s-a)$ [**3**, pp.60, 164 (Ex. 3)]. Any three mutually tangent circles can be considered as having centers A, B, and C and radii s-a, s-b, s-c (in the case of external contact), or else s, s-c, s-b (internal contact).



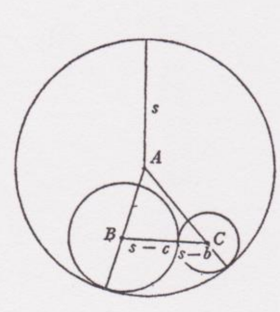


Figure 3. *external contact* [**2**, pp. 7]   Figure 4. *internal contact* [**2**, pp. 7]

In the case of *external contact*, let $1/k_1 = r$, $1/b_2 = s\text{-}a$, $1/b_3 = s\text{-}b$, and $1/b_4 = s\text{-}c$. Hence, we have just defined the radii of the three mutually tangent circles with curvatures b_2, b_3, and b_4, as well as the radius of the *incircle* of the triangle ABC with curvature k_1. Expanding the right hand side of (1.1) we see that,

$$(\sum b_i)^2 = \sum b_i^2 + 2\sum b_i b_j. \qquad (1.2^*)$$

We know that $r^2 = (s\text{-}a)(s\text{-}b)(s\text{-}c)/s$. Let's invert this equation to obtain,

$$1/r^2 = s/(s\text{-}a)(s\text{-}b)(s\text{-}c).$$

We know that $k_1 = 1/r$ by definition. Squaring each side we get $k_1^2 = 1/r^2$. Hence,

$$k_1^2 = s/(s\text{-}a)(s\text{-}b)(s\text{-}c), \qquad \text{(transitivity)}$$

$$= s\,(b_2 b_3 b_4). \qquad \text{(substitution)}$$

Since s is the semiperimeter of triangle ABC, $s = (s\text{-}a) + (s\text{-}b) + (s\text{-}c)$. Hence, by substitution and algebra,

$$k_1^2 = [(s\text{-}a) + (s\text{-}b) + (s\text{-}c)]\, b_2 b_3 b_4 = (1/b_2 + 1/b_3 + 1/b_4)\, b_2 b_3 b_4 = b_2 b_3 + b_2 b_4 + b_3 b_4.$$

Therefore,

$$k_1^2 = b_2 b_3 + b_2 b_4 + b_3 b_4. \qquad \text{(i)}$$

In the case of *internal contact* Let $1/k_1=r_e$, $1/b_2=-s$, $1/b_3=s-c$, and $1/b_4=s-b$. (The minus sign is used to specify the internal contact.) Hence, we have just defined the radii of the three mutually tangent circles with curvatures b_2, b_3, and b_4, as well as the radius of the first *excircle* of the triangle ABC with curvature k_1.

We know that $r_e^2 = s(s-b)(s-c)/(s-a)$. Let's invert this equation to obtain,

$$1/r_e^2 = (s-a)/-s(s-b)(s-c)$$

We know that $k_1= 1/r_e$ by definition. Squaring each side we get $k_1^2 = 1/r_e^2$. Hence,

$$k_1^2 = (s-a)/-s(s-b)(s-c), \qquad \text{(transitivity)}$$

$$= (s-b-c)\ (b_2b_3b_4). \qquad \text{(substitution)}$$

Hence, by substitution and algebra,

$$k_1^2 = (s-b-c)\ b_2b_3b_4 = (1/b_2 + 1/b_3 + 1/b_4)\ b_2b_3b_4 = b_2b_3 + b_2b_4 + b_3b_4.$$

Therefore,

$$k_1^2 = b_2b_3 + b_2b_4 + b_3b_4. \qquad \text{(i)}$$

Similarly, if we instead choose the three mutually tangent circles with bends k_2, k_3, and k_4, and the incircle (or excircle) b_1, we would find that $k_3k_4+k_2k_4+k_2k_3=b_1^2$ (which becomes useful in 1.5). Notice that, initially, we were able to choose the three mutually tangent circles arbitrarily. Hence, we can permute the subscripts 1, 2, 3, 4 to obtain: (ii) $k_2^2 = b_3b_4+b_4b_1+b_1b_3$, (iii) $k_3^2 = b_1b_4+b_4b_2+b_2b_1$, and (iv) $k_4^2 = b_3b_1+b_1b_2+b_2b_3$. Adding together the equations (i), (ii), (iii), and (iv), we get

$$k_1^2 + k_2^2 + k_3^2 + k_4^2 = b_3b_4 + b_2b_4 + b_2b_3 + b_3b_4 + b_4b_1 + b_1b_3 + b_1b_4 + b_4b_2 + b_2b_1 + b_3b_1 + b_1b_2 +b_2b_3, \qquad (1.3)$$

which can also be written as,

$$\sum k_i^2 = 2 \sum b_i b_j. \qquad (1.4^*)$$

Similarly, as we hinted in the previous paragraph, if we selected three mutually tangent circles with curvatures k_i and the respective incircle b_i, we get

$$\sum b_i^2 = 2 \sum k_i k_j. \qquad (1.5^*)$$

Therefore, combining (1.2), (1.4) and (1.5) we obtain,

$$(\sum b_i)^2 = \sum b_i^2 + 2\sum b_i b_j = \sum b_i^2 + \sum k_i^2 = 2 \sum k_i k_j + \sum k_i^2 = (\sum k_i)^2. \qquad (1.6^*)$$

Hence,

$$\sum b_i = \sum k_i. \qquad (1.7)$$

We want to show that $\sum b_i^2 = \sum k_i^2$.

Consider,

$$-b_1^2 + (b_2 + b_3 + b_4)^2 = -b_1^2 + b_2^2 + b_3^2 + b_4^2 + 2\,(b_2b_3 + b_2b_4 + b_3b_4)$$

$$= -b_1^2 + b_2^2 + b_3^2 + b_4^2 + 2k_1^2$$

$$= -b_1^2 + k_1k_3 + k_1k_4 + k_3k_4 + k_1k_2 + k_1k_4 + k_2k_4 + k_1k_3 + k_1k_2 + k_2k_3 + 2k_1^2$$

$$= 2k_1\,(k_2 + k_3 + k_4) + 2k_1^2$$

$$= 2k_1\,(k_1 + k_2 + k_3 + k_4)$$

$$= 2k_1 \sum k_i$$

$$= 2k_1 \sum b_i.$$

Thus,

$$-b_1 + b_2 + b_3 + b_4 = 2k_1.$$

Adding four such equations after squaring each side, we obtain

$$\sum b_i^2 = \sum k_i^2. \qquad (1.8)$$

Therefore, by (1.6) and (1.8), $2\sum b_i^2 = \sum b_i^2 + \sum k_i^2 = (\sum b_i)^2$, which completes the proof.

$\square$

An immediate consequence of the Descartes' circle theorem is that given the curvature of three mutually tangent circles, we can solve for the curvatures of the two circles that are mutually tangent to all three original circles. From this new collection of four mutually tangent circles, we can then arbitrarily choose three of them, and solve for the curvature of two new circles that are mutually tangent to this selection of three circles. This process can be repeated indefinitely, which is the process used for constructing an Apollonian Gasket.

1. Jeffrey C. Lagarias, Colin L. Mallows and Allan R. Wilks, *Beyond the Descartes Circle Theorem*, The American Mathematical Monthly, Vol. 109, No. 4 (Apr., 2002), pp. 338-361 Published by: Mathematical Association of America, Article Stable http://www.jstor.org/stable/2695498.

2. H. S. M. Coxeter, *The problem of Apollonius*, American Mathematical Monthly, Vol. 75, No. 1 (Jan., 1968), pp. 5-15.

3. H. S. M. Coxeter and S. L. Greitzer, Geometry Revisited, New Math. Library, Vol. 19, New York and Syracuse, 1967.

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(1) 2. H. S. M. Coxeter, The problem of Apollonius, American Mathematical Monthly, Vol. 75, No. 1 (Jan., 1968), pp. 5-15.

(2) H. S. M. Coxeter and S. L. Greitzer, Geometry Revisited, New Math. Library, Vol. 19, New York and Syracuse, 1967.

(3) Jeffrey C. Lagarias, Colin L. Mallows and Allan R. Wilks, Beyond the Descartes Circle Theorem, The American Mathematical Monthly Vol. 109, No. 4 (Apr., 2002), pp. 338-361 Published by: Mathematical Association of America, Article Stable http://www.jstor.org/discover/10.2307/2695498?uid=3739960&uid=2134&uid=2&uid=70&uid=4&uid=3739256&sid=21101973433083

(4) D. Mackenzie, A Tisket, a Tasket, an Apollonian Gasket, American Scientist, V. 98, no. 1, (2010) http://www.americanscientist.org/issues/pub/a-tisket-a-tasket-an-apollonian-gasket/1