# Compass and Straightedge Constructions II: Regular Polygons

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## I. Origin/History of the Golden Ratio

The golden ratio is a phenomenal number which has prevalence not only in a wide variety of mathematical practice, but in other pedigrees as well. Recognized potentially in 400 BC, this number has gained the attention of countless mathematicians and individuals of many different disciplines. This famous number is most common represented by the Greek letter phi, φ (Figure 1) and its approximation is 1.6180339887... (Figure 2).

$$\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi$$

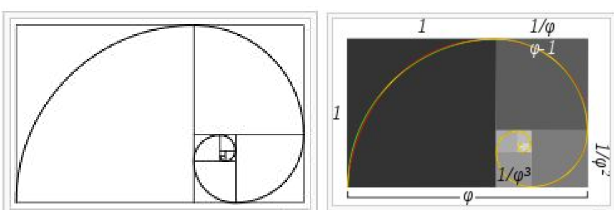
Figure 1. Definition of the Golden Ratio. "Golden Ratio." *Wikipedia: The Free Encyclopedia.* Wikimedia Foundation, Inc. 27 May 2013. Web. 27 May 2013. <http://en.wikipedia.org/wiki/Golden_ratio>

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$$

Figure 2. Value of the Golden Ratio. "Golden Ratio." *Wikipedia: The Free Encyclopedia.* Wikimedia Foundation, Inc. 27 May 2013. Web. 27 May 2013. <http://en.wikipedia.org/wiki/Golden_ratio>

It is speculated that the oldest example of the golden ratio lies in the Parthenon, a Greek temple built in 447BC. The height of the Parthenon's columns as well as the width can be shown to illustrate the golden ratio in design, however, whether this was done by chance or on purpose is a still debated subject. Euclid (325BC – 265BC) provided the first known written definition of the golden ratio, although at the time he referred to it as the extreme and mean ratio. Since then, countless mathematicians have explored and researched this ratio.

Why is the golden ratio so special? At first, this ratio was constructed and researched because of its very common occurrence in geometry. Examples include the construction of a regular pentagon, hexagon and many other geometric shapes. It also has interesting ties with sequences such as the Fibonacci sequence (Figure 3).




An approximation of the golden spiral created by drawing circular arcs connecting the opposite corners of squares in the Fibonacci tiling; this one uses squares of sizes 1, 1, 2, 3, 5, 8, 13, 21, and 34.

Approximate and true golden spirals: the green spiral is made from quarter-circles tangent to the interior of each square, while the red spiral is a golden spiral, a special type of logarithmic spiral. Overlapping portions appear yellow. The length of the side of a larger square to the next smaller square is in the golden ratio.

Figure 3. The Golden Ratio and the Fibonacci Sequence. "Golden Ratio." *Wikipedia: The Free Encyclopedia.* Wikimedia Foundation, Inc. 27 May 2013. Web. 27 May 2013. <http://en.wikipedia.org/wiki/Golden_ratio>

However, what really makes this number special is not its occurrence in math, but in biology, art, music, history, architecture and even psychology. The golden ratio appears in everything from Leonardo's paintings to the acoustic scale to the arrangements of branches along the stems of plants.

## II. List of Definitions, Lemmas and Theorems needed to construct the Golden Ratio and the regular polygons

Definition 1 (The Golden Triangle)

A golden triangle is an isosceles triangle in which the ratio of the length of each leg to the length of the base is equal to the golden ratio.

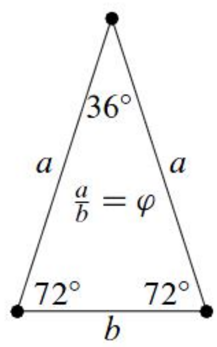



Figure 4. The Golden Triangle. Lee, Jack. *Axiomatic Geometry*.

Lemma 1

Given a line segment, we can construct an equilateral triangle on either side of it.

**Proof:** Let A and B be any two points. Construct the circle with center A and passing through B (we'll call it C1) and the circle with center B and passing through A (we'll call it C2). If we let r be the radius of C1 and s be the radius of C2, we know that |AB| = r and |BA| = s. Since |AB| = |BA|, we know r = s. This implies that |AB| < r + s; r < |AB| + s; and s < |AB| + r. By Theorem 14.10, this means that C1 and C2 intersect at two points, one on each side of <-AB->. Let C and C' be these points of intersection. Since C is a point on C1 and C2, we know that |AC| = |AB| = |BC| and thus, △ABC is an equilateral triangle. We can say the same thing for △ABC'.

Lemma 2

Given any line segment, we can construct its midpoint

**Proof:** Let A and B be any two points. Start by constructing an equilateral triangle on either side of –AB-: △ABC and △ABC'. Thus, we know that |AC| = |BC| = |BC'| = ||AC'|, which by definition means that ACBC' is a rhombus. By Corollary 9.18 this means that ACBC' is a parallelogram and by Corollary 9.7, this means that ACBC' is convex. This, by Theorem 9.4, means that –AB- and –CC'- (the diagonals of our rhombus) have a point in common. Let's call this point M.

We will show that M is the midpoint of –AB- by showing that |AB| = |BM|. To do this, let's examine the triangles △CBC' and △CAC'. We know that –CB- is congruent to –CA-, -C'B- is congruent to –C'A- and that –C'C- is congruent to itself. Thus, by Theorem 5.12, we know that △CBC' is congruent to △CAC'. Thus, |AM| = |BM| and thus M is the midpoint of –AB-.

## Lemma 3

### Given any segment, -AB-, we can construct a perpendicular line to <-AB-> that contains B

**Proof:** Let –AB- be a segment. Lemma 1 guarantees the existence of a midpoint, M, on –AB-. Now, extend –AB- to create the ray –AB->. If we draw a circle centered at B passing through M, we se that since <-AB-> is a secant line to our circle, then <-AB-> must intersect it at two points. Let M' be the other point of intersection. We know that M, B, and M' are collinear since they all lie on <-AB->. Thus, either B * M * M', B * M' * M or M * B * M' (Theorem 3.9).

Case I: B * M * M'

By Theorem 3.22e, this case implies that |BM'| > |BM|. However, since these are both distances from the center of our circle to a point on our circle, this inequality cannot be true. Thus this case is not possible.

Case II: B * M' * M

The same argument as Case I applies here.

Thus, we know that M * B * M'.

Now, construct an equilateral triangle, △MDM' on one side of –MM'-. Now, since M * B * M', we can see that △MDM' can be split by the chord –BD-. If we examine △MBD and △M'BD, we can see that –MB- and –M'B- are congruent (since they both are radiuses of our circle), that –MD- is congruent to –M'D- (by the definition of an equilateral triangle) and that –BD- is congruent to itself. Thus, by Theorem 5.12, these two triangles are congruent and thus, we know that angle(MBD) is congruent to angle(M'BD) We can also see that these angles form a linear pair, so by Corollary 4.15, we know that they are both right angles. Thus, if we extend –BD- at both ends to form the line <-BD->, we can see that this line is perpendicular to –AB- and passing through B.

## Theorem 14.6 (Line-Circle Theorem)

### Suppose C is a circle and $\ell$ is a line that contains a point in the interior of C. Then $\ell$ is a secant line for C, and thus there are exactly two points where $\ell$ intersects C.

## Theorem 14.10 (Two Circles Theorem)

Suppose C and D are two circles. If either of the following conditions is satisfied, then there exist exactly two points where the circles intersect, one on each side of the line containing their centers.

(a) The following inequalities all hold:

$d < r + t$, $s < r + d$, $r < d + s$,

where r and s are the radii of C and D, respectively, and d is the distance between
their centers.

(b) D contains a point in the interior of C and a point in the exterior of C.

Lemma 14.41

Every equilateral polygon inscribed in a circle is regular.

## III. The Golden Ratio

### III.a. Diagram

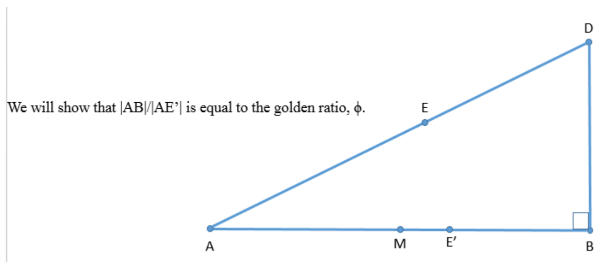



Figure 5. Constructing the Golden Ratio

### III.b. Construction and proof

Now, we will construct the golden ratio. First, construct any two points A and B. Next, we use Lemma 2 to construct the midpoint of –AB-, call this point M. After this, we use Lemma 3 to construct a line, *l*, perpendicular to <-AB-> through the point B. Then, we draw a circle centered at B passing through M and take D to be the point at which this circle intersects *l*. We know D exists by Theorem 14.6.

If we connect A and D, then we know that ∆ABD is a right triangle, and therefore, Corollary 5.17 guarantees that |AD| > |BD|. Because of this, we can draw the circle, $l_2$ with center D passing through B, and let E be the point where $l_2$ meets the interior of –DA-. Again, by Theorem 14.6, we can draw the circle $l_3$ with center A and passing through E, and let E' be the point where $l_3$ intersects –AB->.

We will show that |AB|/|AE'| is equal to the golden ratio, ϕ. Note that |BD| = |BM| = ½ * |AB| by construction. Further, the Pythagorean Theorem states that $|AD|^2 = |AB|^2 + |BD|^2$ which we can re-write as $|AD|^2 = |AB|^2 + ¼*|AB|^2$ or |AD| = ½ * √5 * |AB|. Note, also that |DE| = |DB| = ½ * |AB| and that |AE| = |AE'| by construction. In addition to this, since E lies in the interior of –AD-, then A *

E * D and thus by Theorem 3.8, |AE| + |DE| = |AD|. By using all of this information, we can say that:

$$|AE'| = |AE| = |AD| - |DE| = ½ * \sqrt{5} * |AB| - ½ * |AB| = \frac{\sqrt{5}-1}{2} * |AB| = (\phi - 1)*|AB|$$

And by equation 12.23, we know that:

$$(\phi - 1) = \frac{1}{\phi}$$

So thus, $|AE'| = \frac{1}{\phi} * |AB|$ which by using simple algebra can be rewritten as:

$$\frac{|AB|}{|AE'|} = \phi$$

Thus, we have constructed the Golden Ratio.

## IV. The Regular Pentagon

### IV.a. Diagram

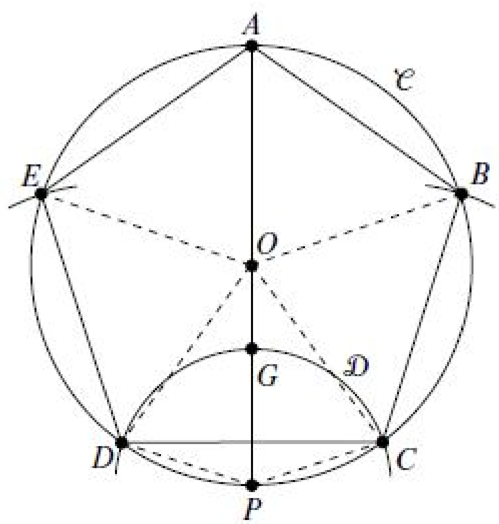



Figure 6. Construction of the Regular Pentagon. Lee, Jack. *Axiomatic Geometry*.

### IV.b. Construction and proof

Construct an arbitrary point O and construct a circle C1 with center O. Let A be any point on C1. Draw the line <-AO-> and let P be the point where <-AO-> intersects C1. We know that P exists by Theorem 14.6. Further, since A and P are points on C1 and since O is a point in the interior of C1, we know that A * O * P. Let G be the point on the interior of –PO- such that $\frac{|PO|}{|PG|} = \phi$, the golden ratio.

We know we can construct this point from what we showed in the previous section. Note here that O * G * P which implies that A * G * P, Thus, A is not an interior point of –GP- (this will become important soon).

Now, construct the circle C2 centered at P and passing through the point G. Since P is an interior point of C2, G is a point on C2, and A is not an interior point of –GP-, we know that A must lie in the exterior of C2 by Theorem 14.4. Thus, C1 contains a point in the interior of C2 (P) and a point in the exterior of C2 (A). By Theorem 14.10, this means that C1 and C2 intersect at two points. Let's call these points C and D. This theorem also guarantees that C and D lie on opposite sides of <-OP->.

We note here that since C and D are points on C1, the maximum value |CD| can take on would be d2, the diameter of C2. Further, we note that since O * G * P, then |OP| > |GP| (Theorem 3.22e). Thus, if we let r1 be the radius of C1 and r2 be the radius of C2, then we know that r1 > r2. This means that if d1 is the diameter of C1, then d1 > d2.

Now, if we construct the line <-OC->, we know that <-OC-> intersects C1 at two points (C and C') by Theorem 14.6. We know that –CC'- is a diameter of C1 and thus |CC'| = d1. Thus, we know that since d1 > d2 and since |CD| can be no larger than d2, then |CC'| > |CD|. Thus, by Corollary 3.37, we can construct a point, D' in the interior of –CC'- such that |CD'| = |CD|. We know, then, that C * D' * C'. Now let's construct the circle centered at C and passing through D and call it C3. Since |CD| = |CD'|, we know that D' lies on C3. Also, we know that C lies in the interior of C3. Further, since C * D' * C', we know that C' does not lie in the interior of –CD'- we know that C' must lie in the exterior of C3 by Theorem 14.4. Therefore, C1 contains a point in the interior of C3 (C) and a point in the exterior of C3 (C'). Hence, by Theorem 14.10, we know that C1 and C3 must intersect at two points. Let's call the second point of intersection B.

We use this same method to construct a circle, C4, centered at D and passing through C. By the same argument as above, we know that C4 intersects C1 at two points so let the second point of intersection be E. We note here that since –CD- is a radius of both C3 and C4, then these circles are the same size. Thus, since –CB- is a radius of C3 and since –DE- is a radius of C4, then |CB| = |DE| = |CD|.

Finally, construct the pentagon ABCDE which is inscribed in C1 by construction and we would like to show that ABCDE is equilateral.

Since –PC- and –PG- are radii of the circle C2, we know that |PC| = |PG|. Similarly, we know that |PO| = |CO|. Thus, we know that $\frac{|PO|}{|PC|} = \frac{|PO|}{|PG|} = \phi$. It follows, then, that △POC is a golden triangle and thus m<POC = 36. Similarly, m<POD = 36. Since D and C lie on opposite sides of <-OP-> then we know that <POC and <POD are adjacent angles. Thus

m<COD = m<POC + m<POD = 36 + 36 = 72.

Now, let's consider △BOC, △DOE and △COD. We know that |EO| = |DO| = |CO| = |BO| since they are all radii of C1. Further, we have already stated that |ED| = |CD| = |BC|. Thus, we know, by the SSS congruence, that △BOC is congruent to △DOE is congruent to △COD. Therefore, m<BOC = m<DOE = 72. If we apply the Linear Triple Theorem to <AOB, <BOC, and <POC, then we would see that:

m<AOB = 180 – m<BOC – m<POC = 180 – 72 – 36 = 72.

We use the same argument to show that m<AOE = 72 as well. Once we have done this, we can use the SAS Congruence Theorem to prove that △AOB is congruent to △AOE and these two triangles are congruent to △BOC, △COD, and △DOE.

Hence, we know that |AB| = |BC| = |CD| = |DE| = |EA| which means that ABCDE is equilateral. Thus, by Lemma 14.41, it is a regular pentagon.

## V. The Regular Hexagon

### V.a. Diagram

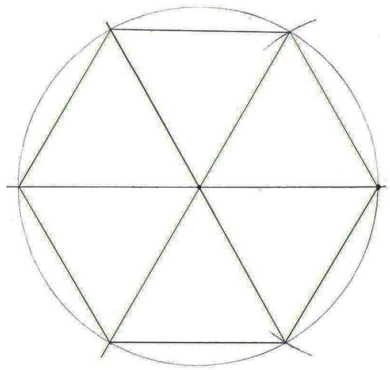


Figure 7. Construction of the Regular Hexagon. "Constructing Regular Polygons." *The University of Georgia Mathematics Education Program.* The University of Georgia. Web. 27 May 2013. <http://jwilson.coe.uga.edu/EMAT6680Fa08/Broderick/essay2/essay2.html>

### V.b. Construction and proof

Let ***C*** be a given circle. We wish to inscribe a regular hexagon. Let O be the center of ***C***, and let A be a point on ***C***. Let ***C1*** be the circle centered at A with radius AO. Segment OA is a radius for both of these circles, and this distance is also the distance between their centers, so by Theorem 14.10, circles ***C*** and ***C1*** intersect in exactly two points. Let these points be called B and F.

Because points A and B are on ***C***, AO and BO have the same length. Because points B and O are on ***C1***, AO and AB have the same length. Therefore, all three sides of triangle ABO have the same length, so it is an equilateral triangle. By the same reasoning, triangle AFO is an equilateral triangle.

Let us construct circle ***C2*** centered at B with radius BO. This circle intersects ***C*** in two points. One of these points is A, and the other point lies on the opposite side of line BO as A, so it is distinct from points A, B, and F. Let us call this point C. By the same reasoning that we have developed previously, triangle BOC is equilateral.

Because we know that ray OF * OA * OB and ray OA * OB * OC and measure angle FOA + measure angle AOB + measure angle BOC = 60 + 60 + 60 =180, we know that rays OF and OC are collinear. More specifically, they are opposite rays, so F*O*C.

Let us examine the line BO. By Theorem 14.6, this line intersects circle ***C*** in two points. One of these points is B. Let us call the other point E. Because O is the center of the circle, we know that B*O*E. Because they are all radii of circle ***C***, BO=EO and CO=FO. Also, because they are vertical angles, angle BOC = angle EOF. Therefore, by SAS, triangles BOC and EOF are congruent equilateral triangles.

Now let us examine line AO. This line also intersects ***C*** at two points. One of these points is A. Let us call the other point D. By very similar reasoning as before, we can show that triangle COD and triangle DOE are both equilateral triangles. Because we have six equilateral triangles, and they each have a side length equal to the radius of the circle, hexagon ABCDEF has six congruent sides, and is thus equilateral. Each angle of ABCDEF has measure of 120 degree, so it is also equiangular. Therefore, we have constructed a regular hexagon inside of circle ***C***.

# Works Cited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