Postulates of Neutral Geometry

Postulate 1 (The Set Postulate). *Every line is a set of points, and there is a set of all points called **the plane**.*

Postulate 2 (The Existence Postulate). *There exist at least three distinct noncollinear points.*

Postulate 3 (The Unique Line Postulate). *Given any two distinct points, there is a unique line that contains both of them.*

Postulate 4 (The Distance Postulate). *For every pair of points A and B, the distance from A to B is a nonnegative real number determined by A and B.*

Postulate 5 (The Ruler Postulate). *For every line ℓ, there is a bijective function $f : \ell \to \mathbb{R}$ with the property that for any two points $A, B \in \ell$, we have $AB = |f(B) - f(A)|$.*

Postulate 6 (The Plane Separation Postulate). *For any line ℓ, the set of all points not on ℓ is the union of two disjoint subsets called the **sides of ℓ**. If A and B are distinct points not on ℓ, then A and B are on the same side of ℓ if and only if $\overline{AB} \cap \ell = \varnothing$.*

Theorems of Neutral Geometry

Theorem 3.1. *Every line contains infinitely many distinct points.*

Corollary 3.2 (Incidence Axiom 4). *Given any line, there are at least two distinct points that lie on it.*

Lemma 3.3 (Ruler Sliding Lemma). *Suppose ℓ is a line and $f : \ell \to \mathbb{R}$ is a coordinate function for ℓ. Given a real number c, define a new function $f_1 : \ell \to \mathbb{R}$ by $f_1(X) = f(X) + c$ for all $X \in \ell$. Then f_1 is also a coordinate function for ℓ.*

Lemma 3.4 (Ruler Flipping Lemma). *Suppose ℓ is a line and $f : \ell \to \mathbb{R}$ is a coordinate function for ℓ. If we define $f_2 : \ell \to \mathbb{R}$ by $f_2(X) = -f(X)$ for all $X \in \ell$, then f_2 is also a coordinate function for ℓ.*

Theorem 3.5 (Ruler Placement Theorem). *Suppose ℓ is a line and A, B are two distinct points on ℓ. Then there exists a coordinate function $f : \ell \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) > 0$.*

Theorem 3.6 (Properties of Distances). *If A and B are any two points, their distance has the following properties:*

(*a*) $AB = BA$.

(*b*) $AB = 0$ *if and only if* $A = B$.

(*c*) $AB > 0$ *if and only if* $A \neq B$.

Theorem 3.7 (Symmetry of Betweenness of Points). *If A, B, C are any three points, then $A * B * C$ if and only if $C * B * A$.*

Theorem 3.8 (Betweenness Theorem for Points). *Suppose A, B, and C are points. If $A * B * C$, then $AB + BC = AC$.*

Theorem 3.9 (Hilbert's Betweenness Axiom). *Given three distinct collinear points, exactly one of them lies between the other two.*

Corollary 3.10 (Consistency of Betweenness of Points). *Suppose A, B, C are three points on a line ℓ. Then $A * B * C$ if and only if $f(A) * f(B) * f(C)$ for every coordinate function $f : \ell \to \mathbb{R}$.*

Theorem 3.11 (Partial Converse to the Betweenness Theorem for Points). *If A, B, and C are three distinct collinear points such that $AB + BC = AC$, then $A * B * C$.*

Theorem 3.12. *Suppose A and B are distinct points. Then*

$$\overleftrightarrow{AB} = \{P : P * A * B \text{ or } P = A \text{ or } A * P * B \text{ or } P = B \text{ or } A * B * P\}.$$

Lemma 3.13. *If $A_1, A_2, \ldots, A_k$ are distinct collinear points, then $A_1 * A_2 * \cdots * A_k$ if and only if $A_k * \ldots A_2 * A_1$.*

Theorem 3.14. *Given any k distinct collinear points, they can be labeled $A_1, \ldots, A_k$ in some order such that $A_1 * A_2 * \cdots * A_k$.*

Theorem 3.15. *Suppose A, B, C are points such that $A * B * C$. If P is any point on $\overleftrightarrow{AB}$, then one and only one of the following relations holds:*

$$P = A, \quad P = B, \quad P = C,$$
$$P * A * B * C, \quad A * P * B * C, \quad A * B * P * C, \quad A * B * C * P.$$

Theorem 3.16. *Suppose* A, B, C, D *are four distinct points. If any of the following pairs of conditions holds, then* $A * B * C * D$*:* $A * B * C$ *and* $B * C * D$*; or* $A * B * C$ *and* $A * C * D$*; or* $A * B * D$ *and* $B * C * D$*. On the other hand, if* $A * B * C * D$*, then all of the following conditions are true:* $A * B * C$*,* $A * B * D$*,* $A * C * D$*, and* $B * C * D$*.*

Lemma 3.17. *If* A *and* B *are two distinct points, then* $\overline{AB} \subseteq \overleftrightarrow{AB}$*.*

Theorem 3.18 (Segment Extension Theorem). *If* $\overline{AB}$ *is any segment, there exist points* $C, D \in \overleftrightarrow{AB}$ *such that* $C * A * B$ *and* $A * B * D$*.*

Lemma 3.19. *Suppose* S *and* T *are sets of points in the plane with* $S \subseteq T$*. Every passing point of* S *is also a passing point of* T*.*

Theorem 3.20. *Suppose* A *and* B *are distinct points. Then* A *and* B *are extreme points of* $\overline{AB}$*, and every other point of* $\overline{AB}$ *is a passing point.*

Corollary 3.21 (Consistency of Endpoints of Segments). *Suppose* A *and* B *are distinct points, and* C *and* D *are distinct points, such that* $\overline{AB} = \overline{CD}$*. Then either* $A = C$ *and* $B = D$*, or* $A = D$ *and* $B = C$*.*

Theorem 3.22 (Euclid's Common Notions for Segments).

(*a*) **Transitive Property of Congruence:** *Two segments that are both congruent to a third segment are congruent to each other.*

(*b*) **Segment Addition Theorem:** *Suppose* A, B, C, A', B', C' *are points such that* $A * B * C$ *and* $A' * B' * C'$*. If* $\overline{AB} \cong \overline{A'B'}$ *and* $\overline{BC} \cong \overline{B'C'}$*, then* $\overline{AC} \cong \overline{A'C'}$*.*

(*c*) **Segment Subtraction Theorem:** *Suppose* A, B, C, A', B', C' *are points such that* $A * B * C$ *and* $A' * B' * C'$*. If* $\overline{AC} \cong \overline{A'C'}$ *and* $\overline{AB} \cong \overline{A'B'}$*, then* $\overline{BC} \cong \overline{B'C'}$*.*

(*d*) **Reflexive Property of Congruence:** *Every segment is congruent to itself.*

(*e*) **The Whole Segment is Greater Than the Part:** *If* A*,* B*, and* C *are points such that* $A * B * C$*, then* $AC > AB$*.*

Lemma 3.23 (Coordinate Representation of a Segment). *Suppose* A *and* B *are distinct points, and* $f : \overleftrightarrow{AB} \to \mathbb{R}$ *is a coordinate function for* $\overleftrightarrow{AB}$*. Then*

$$\begin{aligned} \overline{AB} &= \{P \in \overleftrightarrow{AB} : f(A) \le f(P) \le f(B)\} \qquad \text{if } f(A) < f(B); \\ \overline{AB} &= \{P \in \overleftrightarrow{AB} : f(A) \ge f(P) \ge f(B)\} \qquad \text{if } f(A) > f(B). \end{aligned}$$

Theorem 3.24. *If* A, B, C *are points such that* $A * B * C$*, then the following set equalities hold:*

(*a*) $\overline{AB} \cup \overline{BC} = \overline{AC}$.

(*b*) $\overline{AB} \cap \overline{BC} = \{B\}$.

Corollary 3.25. *If* $A * B * C$*, then* $\overline{AB} \subseteq \overline{AC}$ *and* $\overline{BC} \subseteq \overline{AC}$*.*

Lemma 3.26. *Let* $\overline{AB}$ *be a segment, and let* M *be a point. The following statements are all equivalent to each other:*

(*a*) M *is a midpoint of* $\overline{AB}$ (*i.e.,* $M \in \operatorname{Int}\overline{AB}$ *and* $MA = MB$).

(*b*) $M \in \overleftrightarrow{AB}$ *and* $MA = MB$*.*

(*c*) $M \in \overline{AB}$ *and* $AM = \frac{1}{2}AB$*.*

Theorem 3.27 (Existence and Uniqueness of Midpoints). *Every segment has a unique midpoint.*

Theorem 3.28. *Every segment contains infinitely many distinct points.*

Theorem 3.29 (Euclid's Postulate 3). *Given two distinct points* O *and* A*, there exists a circle whose center is* O *and whose radius is* OA*.*

Lemma 3.30. *Suppose* A *and* B *are distinct points, and* P *is a point on the line* $\overleftrightarrow{AB}$*. Then* $P \notin \overrightarrow{AB}$ *if and only if* $P * A * B$*.*

Lemma 3.31. *Suppose* A *and* B *are distinct points. Then* $\overline{AB} \subseteq \overrightarrow{AB} \subseteq \overleftrightarrow{AB}$*.*

Lemma 3.32 (Coordinate Representation of a Ray). *Suppose* A *and* B *are distinct points, and* $f\colon \overleftrightarrow{AB} \to \mathbb{R}$ *is a coordinate function for* $\overleftrightarrow{AB}$*. Then*

$$\begin{aligned} \overrightarrow{AB} &= \{P \in \overleftrightarrow{AB} : f(P) \geq f(A)\} \qquad \text{if } f(A) < f(B); \\ \overrightarrow{AB} &= \{P \in \overleftrightarrow{AB} : f(P) \leq f(A)\} \qquad \text{if } f(A) > f(B). \end{aligned}$$

Lemma 3.33 (Representation of a Ray in Adapted Coordinates). *Suppose* A *and* B *are distinct points, and* $f\colon \overleftrightarrow{AB} \to \mathbb{R}$ *is a coordinate function adapted to* $\overrightarrow{AB}$*. If* P *is any point on* $\overleftrightarrow{AB}$*, then* $P \in \overrightarrow{AB}$ *if and only if* $f(P) \geq 0$*, and* $P \in \operatorname{Int}\overrightarrow{AB}$ *if and only if* $f(P) > 0$*.*

Lemma 3.34 (Ordering Lemma for Points). *Suppose* $\overrightarrow{a}$ *is a ray starting at a point* A*, and* B *and* C *are interior points of* $\overrightarrow{a}$ *such that* $AC > AB$*. Then* $A * B * C$*.*

Theorem 3.35 (Segment Construction Theorem). *Suppose* $\overrightarrow{a}$ *is a ray starting at a point* A*, and* r *is a positive real number. Then there exists a unique point* C *in the interior of* $\overrightarrow{a}$ *such that* $AC = r$*.*

Corollary 3.36 (Unique Point Theorem). *Suppose* $\overrightarrow{a}$ *is a ray starting at a point* A*, and* C *and* C' *are points in* $\operatorname{Int}\overrightarrow{a}$ *such that* $AC = AC'$*. Then* $C = C'$*.*

Corollary 3.37 (Euclid's Segment Cutoff Theorem). *If* $\overline{AB}$ *and* $\overline{CD}$ *are segments with* $CD > AB$*, there is a unique point* E *in the interior of* $\overline{CD}$ *such that* $\overline{CE} \cong \overline{AB}$*.*

Theorem 3.38 (Rays with the Same Endpoint). *Suppose* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are rays with the same endpoint.*

- (*a*) *If* A*,* B*, and* C *are collinear, then* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are collinear.*
- (*b*) *If* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are collinear, then they are either equal or opposite, but not both.*
- (*c*) *If* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are opposite rays, then* $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$ *and* $\overrightarrow{AB} \cup \overrightarrow{AC} = \overleftrightarrow{AC}$*.*
- (*d*) $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are equal if and only if they have an interior point in common.*
- (*e*) $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are opposite rays if and only if* $C * A * B$*.*

Theorem 3.39 (Opposite Ray Theorem). *Every ray has a unique opposite ray.*

Theorem 3.40. *Let* $\overrightarrow{AB}$ *be the ray from* A *through* B*. Then* A *is the only extreme point of* $\overrightarrow{AB}$*.*

Corollary 3.41 (Consistency of Endpoints of Rays). *If* A, B *are distinct points and* C, D *are distinct points such that* $\overrightarrow{AB} = \overrightarrow{CD}$*, then* $A = C$*.*

Theorem 3.42. *Suppose* A *and* B *are two distinct points. Then the following set equalities hold:*

- (*a*) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$*.*
- (*b*) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$*.*

Theorem 3.43 (Properties of Sides of Lines). *Suppose* ℓ *is a line.*

- (*a*) *Both sides of* ℓ *are nonempty sets.*
- (*b*) *If* A *and* B *are distinct points not on* ℓ*, then* A *and* B *are on opposite sides of* ℓ *if and only if* $\overline{AB} \cap \ell \neq \varnothing$*.*

Lemma 3.44 (The Y-Lemma). *Suppose* ℓ *is a line,* A *is a point on* ℓ*, and* B *is a point not on* ℓ*. Then every interior point of* $\overrightarrow{AB}$ *is on the same side of* ℓ *as* B*, and* $\overrightarrow{AB} \subseteq \operatorname{CHP}(\ell, B)$*.*

Lemma 3.45 (The X-Lemma). *Suppose* $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$ *are opposite rays, and* ℓ *is a line that intersects* $\overleftrightarrow{AB}$ *only at* O*. Then* $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$ *lie on opposite sides of* ℓ*.*

Theorem 3.46. *Suppose* ℓ *is a line,* A *is a point on* ℓ*, and* B *is a point not on* ℓ*. Then*

$$\overrightarrow{AB} = \overleftrightarrow{AB} \cap \operatorname{CHP}(\ell, B).$$

Theorem 3.47. *If* $S_1, \dots, S_k$ *are convex subsets of the plane, then* $S_1 \cap \dots \cap S_k$ *is convex.*

Theorem 3.48. *Every line is a convex set.*

Theorem 3.49. *Every segment is a convex set.*

Theorem 3.50. *Every open or closed half-plane is a convex set.*

Theorem 3.51. *Every ray is a convex set.*