

## SOLUTIONS FOR THE EXERCISE EB

IN “AXIOMATIC GEOMETRY” BY JOHN LEE

**Notation:**  $\mathcal{P}$  = the set of all points,  $\mathcal{L}$  = the set of all lines. We use capital letters for points and small case letters for lines.

For each of the following statements, do the following:

- Write the statement in symbolic form
- Write its negation in symbolic form
- Rewrite the negations an ordinary English sentence

(a) *Every point lies either on  $\ell$  or on  $m$ .*

- $\forall A \in \mathcal{P}, (A \in \ell) \vee (A \in m)$
- $\exists A \in \mathcal{P}, (A \notin \ell) \wedge (A \notin m)$  OR  $\exists A \in \mathcal{P}, (A \notin \ell \cup m)$
- There exists a point  $A$  such that neither  $\ell$  nor  $m$  contain  $A$ .

(b) *For any three points  $A$ ,  $B$ , and  $C$ , if they are collinear, then there is another point  $D$  that is not equal to  $A$ ,  $B$ , or  $C$ .*

- $\forall A, B, C \in \mathcal{P}, (\exists \ell \in \mathcal{L}, (A \in \ell) \wedge (B \in \ell) \wedge (C \in \ell)) \Rightarrow (\exists D \in \mathcal{P}, (D \neq A) \wedge (D \neq B) \wedge (D \neq C))$ .
- $\exists A, B, C \in \mathcal{P}, \exists \ell \in \mathcal{L}, (A \in \ell) \wedge (B \in \ell) \wedge (C \in \ell) \wedge (\forall D \in \mathcal{P}, (D = A) \vee (D = B) \vee (D = C))$ .
- There exist three collinear points  $A$ ,  $B$ , and  $C$  such that any point  $D$  is equal to one of them.

(c) *For every line  $\ell$ , if  $\ell$  contains three distinct points, then it has points in common with three distinct lines.*

- $\forall \ell \in \mathcal{L},$   
 $(\exists A, B, C \in \mathcal{P}, (A \in \ell) \wedge (B \in \ell) \wedge (C \in \ell) \wedge (A \neq B) \wedge (A \neq C) \wedge (B \neq C)) \Rightarrow$   
 $(\exists m, n, p \in \mathcal{L}, (m \neq n) \wedge (m \neq p) \wedge (n \neq p) \wedge (\ell \cap m \neq \emptyset) \wedge (\ell \cap n \neq \emptyset) \wedge (\ell \cap p \neq \emptyset))$
- $\exists \ell \in \mathcal{L}, \exists A, B, C \in \mathcal{P}, (A \in \ell) \wedge (B \in \ell) \wedge (C \in \ell) \wedge (A \neq B) \wedge (A \neq C) \wedge (B \neq C) \wedge (\forall m, n, p \in \mathcal{L}, (m = n) \vee (m = p) \vee (n = p) \vee (\ell \cap m = \emptyset) \vee (\ell \cap n = \emptyset) \vee (\ell \cap p = \emptyset))$
- There exists a line  $\ell$  containing three distinct points such that it intersects at most two distinct lines.

(d) *There exists at least two distinct points*

- $\exists A, B \in \mathcal{P}, A \neq B$ .
- $\forall A, B \in \mathcal{P}, A = B$ .
- There exists at most one point.

(e) [**Axiom II of Incidence Geometry**] *Given any two distinct points, there is at least one line that contains both of them.*

- $\forall A, B \in \mathcal{P}, (A \neq B) \Rightarrow (\exists \ell \in \mathcal{L}, (A \in \ell) \wedge (B \in \ell)).$
- $\exists A, B \in \mathcal{P}, (A \neq B) \wedge (\forall \ell \in \mathcal{L}, (A \notin \ell) \vee (B \notin \ell)).$
- There exist two distinct points such that no line contains them both.

(f) [**Axiom III of Incidence Geometry**] *Given any two distinct points, there is at most one line that contains both of them.*

- $\forall A, B \in \mathcal{P}, \forall \ell, m \in \mathcal{L}, ((A \neq B) \wedge (A \in \ell) \wedge (B \in \ell) \wedge (A \in m) \wedge (B \in m)) \Rightarrow (\ell = m).$
- $\exists A, B \in \mathcal{P}, \exists \ell, m \in \mathcal{L}, (A \neq B) \wedge (A \in \ell) \wedge (B \in \ell) \wedge (A \in m) \wedge (B \in m) \wedge (\ell \neq m).$   
Equivalently, but shorter:  $\exists A, B \in \mathcal{P}, \exists \ell, m \in \mathcal{L}, A \neq B, \ell \neq m, A, B \in \ell \cap m.$
- There exist two distinct lines that intersect at two distinct points.  
Equivalent: There exist two distinct points and two distinct lines that contain both points.

(g) [**Axioms II, III of Incidence Geometry**] *Given any two distinct points, there is a unique line that contains both of them.*

Here, we have to combine (e) and (f) with the operator  $\wedge$ . It looks long but you need to parse it.

- $\forall A, B \in \mathcal{P}, (A \neq B) \Rightarrow (\exists \ell, (A \in \ell) \wedge (B \in \ell) \wedge (\forall m \in \mathcal{L}, (A \in m) \wedge (B \in m) \Rightarrow (m = \ell)))$
- $\exists A, B \in \mathcal{P}, (A \neq B) \wedge (\forall \ell \in \mathcal{L}, (A \notin \ell) \vee (B \notin \ell) \vee (\exists m \in \mathcal{L}, (A \in m) \wedge (B \in m) \wedge (m \neq \ell)))$
- There exist two distinct points such that either no line contains them or there are at least two distinct lines that contain them.

(h) [**Axiom IV of Incidence Geometry**] *Given any line, there are at least two distinct points that lie on it.*

- $\forall \ell \in \mathcal{L} \exists A, B \in \mathcal{P}, (A \in \ell) \wedge (B \in \ell) \wedge (A \neq B)$
- $\exists \ell \in \mathcal{L} \forall A, B \in \mathcal{P}, (A \notin \ell) \vee (B \notin \ell) \vee (A = B)$   
Equivalently,  $\exists \ell \in \mathcal{L} \forall A, B \in \mathcal{P}, (A, B \in \ell) \Rightarrow (A = B)$
- There exists a line with at most one point that lies on it.

**Note:** for the rest of the problems, we'll stop writing " $\in \mathcal{P}$ " and " $\in \mathcal{L}$ " - it is understood implicitly from the notation that points are in  $\mathcal{P}$  and lines are in  $\mathcal{L}$ .

(i) *Given any line, there is at least one point that does not lie on it.*

- $\forall \ell \exists A, A \notin \ell.$
- $\exists \ell \forall A, A \in \ell.$
- There exists a line that contains all the points

(j) *There exists a line  $\ell$  such that for every point  $S$ ,  $S$  lies on  $\ell$ .*

- $\exists \ell \forall A, A \in \ell$
- $\forall \ell \exists A, A \notin \ell$
- Given any line, there exists a point that does not lie on it.

(k) *there exists a point  $A$  such that  $A$  does not lie on any line.*

- $\exists A \forall \ell, A \notin \ell$
- $\forall A \exists \ell, A \in \ell$
- Given any point there exists a line that contains it.

(l) *There exist three distinct points that do not lie on any one line.*

- $\exists A, B, C, (A \neq B) \wedge (A \neq C) \wedge (B \neq C) \wedge (\forall \ell, (A \notin \ell) \vee (B \notin \ell) \vee (C \notin \ell))$
- $\forall A, B, C, (A = B) \vee (A = C) \vee (B = C) \vee (\exists \ell, (A \in \ell) \wedge (B \in \ell) \wedge (C \in \ell))$
- Any three distinct points are collinear.

(m) *In any triangle  $ABC$ , if  $AB = AC$ , then  $\angle B = \angle C$*

- $\forall \triangle ABC, (AB = AC) \Rightarrow (\angle B = \angle C)$ .
- $\exists \triangle ABC, AB = AC \wedge \angle B \neq \angle C$ .
- There exists a triangle  $ABC$  such that  $AB = AC$  but  $\angle B \neq \angle C$ .

(n) *Given a line  $\ell$  and a point  $A$  that does not lie on  $\ell$ , there are at least two different lines that contain  $A$  and parallel to  $\ell$*

- $\forall \ell, \forall A, (A \notin \ell) \Rightarrow (\exists m, n, (m \neq n) \wedge (A \in m) \wedge (A \in n) \wedge (m \parallel \ell) \wedge (n \parallel \ell))$
- $\exists \ell, \exists A, (A \notin \ell) \wedge (\forall m, n, (m = n) \vee (A \notin m) \vee (A \notin n) \vee (m \not\parallel \ell) \vee (n \not\parallel \ell))$
- There exists a line  $\ell$  and a point  $A$  that does not lie on the line  $\ell$  such that there is at most one line which is parallel to  $\ell$  and contains the point  $A$ .

(o) *In any triangle the sum of the measures of any two angles is less than  $180^\circ$*

- $\forall \triangle ABC, (\angle A + \angle B < 180^\circ) \wedge (\angle A + \angle C < 180^\circ) \wedge (\angle B + \angle C < 180^\circ)$ .
- $\exists \triangle ABC, (\angle A + \angle B \geq 180^\circ) \vee (\angle A + \angle C \geq 180^\circ) \vee (\angle B + \angle C \geq 180^\circ)$ .
- There exists a triangle such that the sum of the measures of two of its angles is at least  $180^\circ$ .