MATH 342 Lin Alg II Winter 2004
Practice problems **Name:** ____________________

1. Let a, b be positive real numbers. Using Cauchy-Schwartz inequality, prove that

$$\sqrt{ab} \leq \frac{a+b}{2}$$

Hint. Apply CS inequality to vectors $[\sqrt{a}, \sqrt{b}]$ and $[\sqrt{b}, \sqrt{a}]$.

2. Compute the QR-factorization of $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$.

Answer. $Q = \begin{bmatrix} 0 & \frac{\sqrt{110}}{11} & \frac{\sqrt{11}}{11} \\ \frac{3\sqrt{10}}{10} & -\frac{\sqrt{110}}{110} & \frac{\sqrt{11}}{11} \\ -\frac{\sqrt{10}}{10} & -\frac{3\sqrt{110}}{110} & \frac{3\sqrt{11}}{11} \end{bmatrix}$, $R = \begin{bmatrix} \sqrt{10} & \frac{7\sqrt{10}}{10} & \frac{4\sqrt{10}}{5} \\ 0 & \frac{\sqrt{110}}{10} & -\frac{13\sqrt{110}}{55} \\ 0 & 0 & \frac{4\sqrt{11}}{11} \end{bmatrix}$.

3. Find least square solutions and least square errors for $Ax = b$ in three ways:
   - computing the orthogonal projection of b onto $Col A$
   - constructing the normal equations
   - using the QR-factorization of A

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Answers.

(a) $proj_{Col\,A} b = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix}$;

Normal equations:

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right];$$

Least square solution $\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Since the columns of A are NOT linearly independent, there is no QR decomposition.

(b) $proj_{Col\,A} b = \begin{bmatrix} 4681/1737 \\ 962/1737 \\ 6808/1737 \\ 222/193 \end{bmatrix}$;

Normal equations:

$$\left[\begin{array}{ccc|c} 54 & 0 & 10 & 36 \\ 0 & 27 & -10 & 0 \\ 10 & -10 & 27 & 9 \end{array}\right];$$

Least square solution $\hat{x} = \begin{bmatrix} 1123/1737 \\ 70/1737 \\ 21/193 \end{bmatrix}$

QR decomposition

$$\begin{bmatrix} \frac{2\sqrt{6}}{9} & 0 & \frac{7\sqrt{193}}{1737} \\ \frac{\sqrt{6}}{18} & -\frac{5\sqrt{3}}{9} & -\frac{28\sqrt{193}}{1737} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{9} & -\frac{20\sqrt{193}}{1737} \\ \frac{\sqrt{6}}{18} & -\frac{\sqrt{3}}{9} & \frac{40\sqrt{193}}{579} \end{bmatrix} \begin{bmatrix} 3\sqrt{6} & 0 & \frac{5\sqrt{6}}{9} \\ 0 & 3\sqrt{3} & -\frac{10\sqrt{3}}{9} \\ 0 & 0 & \frac{\sqrt{193}}{3} \end{bmatrix}$$

4. Find A^{10} for $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$.

Answer.
$A = \frac{1}{2} \begin{bmatrix} 6^{10} + 4^{10} & 6^{10} - 4^{10} \\ 6^{10} - 4^{10} & 6^{10} + 4^{10} \end{bmatrix}$.

5. Make a change of variable $x = Py$ which transform quadratic forms below into quadratic forms without cross-product terms. Give P and new quardatic forms. Classify quadratic forms.

   (a) $6xy + 8yz$.

   (b) $-5x^2 + 4xy - 2y^2$.

Answer.

(a) New form: $5(y_1^2 - z_1^2)$

$$P = \begin{bmatrix} -4/5 & 3/\sqrt{50} & -3/\sqrt{50} \\ 0 & 5/\sqrt{50} & 5/\sqrt{50} \\ -3/5 & 4/\sqrt{50} & 4/\sqrt{50} \end{bmatrix}.$$

The form is indefinite.

(b) New form: $-6x_1^2 - y_1^2$

$$P = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

The form is negative definite.

6. Let A be a symmetric 2×2 matrix. Prove that eigenvalues of A are real.

7. Let $\mathbf{P}_3$ have the inner product given by evaluation at $(-2, -1, 1, 2)$. Find the best approximation to $t^3 + 17t$ by polynomials in the span of 1, t, $t^2 - 81t + 100$.

Answer.

$\frac{204}{10}t$

8. Find the maximal and minimal values of the function $f(x, y, z) = x + y + z$ on the unit sphere $x^2 + y^2 + z^2 = 1$.

Answer. Let $Q = (x + y + z)^2$. The no cross product term form of Q is $3t^2$. Thus, the maximal value of Q is 3. We conclude that

$$\max_{x^2+y^2+z^2=1} f(x, y, z) = \sqrt{3}, \quad \min_{x^2+y^2+z^2=1} f(x, y, z) = -\sqrt{3}.$$