**MATH 342 Lin Alg II Winter 2004**
**Midterm** **Name:** ______________________

No book, notes or calculators are alllowed. Show ALL your work.

1. Let $A = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix}$

(a) Find eigenvalues and eigenvectors of $A$.

**Solution.** Eigenvalues: $\lambda_1 = -\frac{1}{2} - \frac{1}{2}i$, $\lambda_2 = -\frac{1}{2} + \frac{1}{2}i$;

Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

(b) Describe geometrically the linear transformation of the plane defined by the matrix $A$.

**Solution.** $A = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} =$
$\begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \cos(3\pi/4) & -\sin(3\pi/4) \\ \sin(3\pi/4) & \cos(3\pi/4) \end{bmatrix}.$
Thus, $A$ is rotation by $3\pi/4$ followed by scaling by $1/\sqrt{2}$.

(c) Compute $A^8$. Describe geometrically the linear transformation defined by $A^8$.

**Solution.** $A^8 = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}^8 \begin{bmatrix} \cos(3\pi/4) & -\sin(3\pi/4) \\ \sin(3\pi/4) & \cos(3\pi/4) \end{bmatrix}^8 =$

$\begin{bmatrix} 1/16 & 0 \\ 0 & 1/16 \end{bmatrix} \begin{bmatrix} \cos(8 \bullet 3\pi/4) & -\sin(8 \bullet 3\pi/4) \\ \sin(8 \bullet 3\pi/4) & \cos(8 \bullet 3\pi/4) \end{bmatrix} =$

$\begin{bmatrix} 1/16 & 0 \\ 0 & 1/16 \end{bmatrix} \begin{bmatrix} \cos(6\pi) & -\sin(6\pi) \\ \sin(6\pi) & \cos(6\pi) \end{bmatrix} = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/16 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/16 \end{bmatrix}.$

$A^8$ is a scaling by 1/16.

2. Let

$$A = \begin{bmatrix} 0.4 & 0.3 \\ -0.5 & 1.2 \end{bmatrix}$$

be the matrix of the discrete dynamical system $x_{k+1} = Ax_k$ describing relative population of spotted owls and (thousands of) flying squirrels in the old-growth forest of Douglas fir. Show that both owls and squirrels will eventually perish. What should be the initial ratio between the numbers of owls and (thousands of) squirrels so that they perish the fastest? The slowest? Classify the origin (point $(0,0)$) for this dynamical system (an attractor, a repeller or a saddle point).

**Solution.** The characteristic polynomial of $A$ is $(0.4-\lambda)(1.2-\lambda)+0.15 = \lambda^2 - 1.6\lambda + 0.63 = (\lambda-0.7)(\lambda-0.9)$. Thus, eigenvalues are 0.7 and 0.9. Since they both are less than 1, the origin is an attractor for this dynamical system. Since any trajectory is attracted to the origin, $x_k$ will be approaching 0 when $k$ approaches $\infty$ for any initial state $x_0$.

Eigenvectors are

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_1 = 0.7$, and

$v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ corresponding to $\lambda_2 = 0.9$.

Since 0.7 is the smallest in magnitude eigenvalue, the direction of $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the direction of the greatest attraction. Thus, if we start with the equal number of owls and (thousands of) squirrels, they will extinct the fastest. The direction of the other eigenvector, $v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, is the direction of the slowest attraction. Thus, if we start with 3 owls to every 5 (thousands of) squirrels, it will take them the longest time to extinct.

3. Find eigenvalues of the matrix

$$A = \begin{bmatrix} -1 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix}.$$

Is this matrix diagonalizable? Explain.

**Solution.** Since $A$ is upper triangular, its eigenvalues are its diagonal entries: $-1, 2, -2$. $A$ is diagonalizable because its eigenvalues are all distinct.

4. Let

$$A = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}.$$

(a) Diagonalize $A$ if possible.

(b) Compute $A^{289}$.

**Solution.** (a) $ch(A) = \lambda^2 - 1$. Eigenvalues: $1, -1$. Corresponding eigenvectors are $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$A$ is diagonalizable. Namely, $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) $A^{289} = PD^{289}P^{-1} = PDP^{-1} = A = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}$.

5. Let

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

(a) Determine whether $< u_1, u_2, u_3 >$ form an orthogonal basis of $\mathbf{R^3}$. Is this an orthonormal basis?

**Answer.** Orthogonal but not orthonormal.

(b) Let

$$y_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Let $(c_{ij})$ be the coordinates of $y_i$ relative to the basis $< u_1, u_2, u_3 >$, i.e.

$y_1 = c_{11}u_1 + c_{12}u_2 + c_{13}u_3$,

$y_2 = c_{21}u_1 + c_{22}u_2 + c_{23}u_3$,

$y_3 = c_{31}u_1 + c_{23}u_2 + c_{33}u_3$.

Find coefficients $c_{ij}$. Arrange your answer as a $3 \times 3$ matrix $C$ with entires $c_{ij}$.

**Solution.** Using the formula $c_{ij} = \frac{y_i \bullet u_j}{u_j \bullet u_j}$, we find

$$C = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & -1 & 0 \\ 4/3 & 0 & -1/3 \end{bmatrix}.$$

(c) Let $T : \mathbf{R^3} \to \mathbf{R^3}$ be the linear transformation defined by the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Compute the matrix of $T$ relative to the basis $< u_1, u_2, u_3 >$.

**Soltion 1.** Let $\mathcal{B}$ denote the new basis $< u_1, u_2, u_3 >$. Let

$$P = [u_1 u_2 u_3] == \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix}.$$

Then $[T]_\mathcal{B} = P^{-1}AP = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 1/2 & 0 \\ 1/6 & 1/6 & -1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 4/3 \\ 0 & -1 & 0 \\ 2/3 & 0 & -1/3 \end{bmatrix}$.

**Solution 2.** Matrix of the linear transformation $T$ relative to the basis $\mathcal{B} =< u_1, u_2, u_3 >$ equals $[[T(u_1)]_\mathcal{B} \ [T(u_2)]_\mathcal{B} \ [T(u_3)]_\mathcal{B}]] = [[Au_1]_\mathcal{B} \ [Au_2]_\mathcal{B} \ [Au_3]_\mathcal{B}]]$. Observe that

$$Au_1 = y_1;$$

$$Au_2 = y_2;$$

$$Au_3 = y_3.$$

Thus, columns of the matrix $[T]_\mathcal{B}$ are coordinates of $y_1, y_2$ and $y_3$ relative to the basis $< u_1, u_2, u_3 >$. We know how to write $y$'s as linear combinations of $u$'s from (b). Namely,

$$[y_1]_\mathcal{B} = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix}, \quad [y_2]_\mathcal{B} = \begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix}, \quad [y_3]_\mathcal{B} = \begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix}.$$

We therefore get

$$[T]_\mathcal{B} = [[y_1]_\mathcal{B} \ [y_2]_\mathcal{B} \ [y_3]_\mathcal{B}]] = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = C^T = \begin{bmatrix} 1/3 & 0 & 4/3 \\ 0 & -1 & 0 \\ 2/3 & 0 & -1/3 \end{bmatrix}.$$