MATH 342 Lin Alg II Spring 2005
Practice Midterm **Name:** ____________________

1. Let $A = \begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.75 \end{bmatrix}$

   (a) Find eigenvalues and eigenvectors of A.

   (b) Consider a discrete dynamical system $x_{k+1} = Ax_k$. Classify the origin as an attractor, repeller or a saddle point of this dynamical system.

   Let x_0 be the initial state of the dynamical system defined above. Compute the state x_{100} of the system for

   1. $x_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$
   2. $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
   3. $x_0 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

   (d) What is the direction of the greatest repulsion and greatest attraction of the dynamical system above? Estimate the long term growth rate of x_k.

   Solution.

   (a) Eigenvalues: $\lambda_1 = 1.25, \lambda_2 = 0.25$, eigenvectors: $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   (b) Since $\lambda_1 > 1 > \lambda_2$, origin is a saddle point.

   (c)

   1. $x_0 = 3v_1$. Thus, $A^{100}x_0 = A^{100}3v_1 = 3(5/4)^{100}v_1 = \begin{bmatrix} 3(5/4)^{100} \\ 3(5/4)^{100} \end{bmatrix}$
   2. $x_0 = -v_2$. Thus, $A^{100}x_0 = -A^{100}v_2 = -(1/4)^{100}v_2 = \begin{bmatrix} \frac{1}{4^{100}} \\ -\frac{1}{4^{100}} \end{bmatrix}$
   3. $x_0 = 2v_1 + 3v_2$. Thus, $A^{100}x_0 = A^{100}(2v_1 + 3v_2) = 2A^{100}v_1 + 3A^{100}v_2 = 2(5/4)^{100}v_1 + 3(1/4)^{100}v_2 = \begin{bmatrix} 2(5/4)^{100} - 3(1/4)^{100} \\ 3(5/4)^{100} + 3(1/4)^{100} \end{bmatrix}$

   (d) The direction of the greatest repulsion is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and of greatest attraction is $v_2 == \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   When k is sufficiently large, we have $x_{k+1} \simeq \lambda_1 x_k = 1.25x_k$. Thus, the long term growth is 25%.

2. Let A be a square matrix of size $n \times n$ with *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ be the corresponding eigenvectors. Show that $v_1, v_2, \ldots v_n$ are linearly independent.

   Solution. See notes from class or section 5.1 in the book.

3. Let $T : \mathbf{R^3} \to \mathbf{R^3}$ be a linear transformation defined by the matrix $A = \begin{bmatrix} -1 & 0 & -3 \\ 3 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix}$, $T(x) = Ax$. Determine whether there exists a basis of $\mathbf{R^3}$ relative to which the matrix of T is diagonal.

   Solution.

   Yes. The basis is $\mathcal{B} =< [3, -3, 1], [-1, 1, 0], [0, 1, 0] >$; $\mathcal{B}$ consists of three linearly independent eigenvectors of A.

4. Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$. Diagonalize A if possible.

   Solution.

   $ch(A) = x^3 - 3x^2 + 4$. Eigenvalues: $\lambda_1 = -1$ with multiplicity 1, $\lambda_2 = 2$ with multiplicity 2. Solving the homogeneous system $(A - 2I)x = 0$, we see that it has only one free variable. Thus, the nullspace has dimension 1. Therefore, the eigenspace of λ_2 has dimension 1 (only one linearly independent eigenvector). Thus, A is not diagonalizable.

5. a). Let $T : \mathbf{P^3} \to \mathbf{P^2}$ be a linear transformation given by the differential: for a polynomial $p(t)$,

$$T(p(t)) = p'(t).$$

Compute the matrix of this linear transformation relative to the bases $< 1, t, t^2 >$ of $\mathbf{P^2}$ and $< 1, t, t^2, t^3 >$ of $\mathbf{P^3}$. What is the rank of the differential as a linear transformation?

Solution.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Rank $[T] = 3$.

b). Let $S : \mathbf{P^2} \to \mathbf{P^3}$ be a linear transformation given by the formula

$$S(p(t)) = \int_0^t p(s)ds$$

Compute the matrix of S relative to the same bases as in (a).

Solution.

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

c). **Without doing matrix multiplication**, answer the following question. Let A be the matrix of T (from (a)) and let B be the matrix of S (from (b)). Find AB.

Solution. Integration followed by differentiation returns the same function we start with. Thus, $AB = I_3$.

6. Let $T : \mathbf{R^3} \to \mathbf{R^3}$ be a linear transformation defined by the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$.

and let $\mathcal{B} =< b_1, b_2, b_3 >$ where $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Compute the matrix of T relative to the basis $\mathcal{B}$.

Solution.

Let

$$P = [b_1\, b_2\, b_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$[T]_B = P^{-1}AP = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 2 & 6 \\ -1 & -1 & 0 \end{bmatrix}$$

7. For each matrix below

   - Draw a few typical trajectories of the discrete dynamical system defined by the matrix A.

   - Compute A^{10}.

   (a) $A = \begin{bmatrix} -4 & -4 \\ 10 & 8 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$

   Solution.

   (a) Eigenvalues if A are complex numbers $2-2i, 2+2i$. Thus, we take A to the form $A = PCP^{-1}$ where C is a rotation/dilation matrix similar to A. For this, we pick eigenvalue $2-2i$, compute the corresponding eigenvector, which is $\begin{bmatrix} 2 \\ -3+i \end{bmatrix}$, and set

$$P = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}.$$

   We have $A = PCP^{-1}$. Therefore, $A^{10} = PC^{10}P^{-1}$.

   First, we compute C^{10}.

$$C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}.$$

   Thus,

$$C^{10} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}^{10} \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}^{10} = \begin{bmatrix} 2^{15} & 0 \\ 0 & 2^{15} \end{bmatrix} \begin{bmatrix} \cos(10\pi/4) & -\sin(10\pi/4) \\ \sin(10\pi/4) & \cos(10\pi/4) \end{bmatrix} =$$

$$\begin{bmatrix} 2^{15} & 0 \\ 0 & 2^{15} \end{bmatrix} \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 2^{15} & 0 \\ 0 & 2^{15} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -2^{15} \\ 2^{15} & 0 \end{bmatrix}.$$

   Finally, $A^{10} = PC^{10}P^{-1} = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2^{15} \\ 2^{15} & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 3/2 & 1 \end{bmatrix} = \begin{bmatrix} -3 \bullet 2^{15} & -2^{16} \\ 5 \bullet 2^{15} & 3 \bullet 2^{15} \end{bmatrix} = 2^{15} \begin{bmatrix} -3 & -2 \\ 5 & 3 \end{bmatrix}$.

   Sample trajectory : outward elliptical spiral with main axes along the vectors which make matrix P: $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$.

Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. So, A is diagonalizable, and the diagonal form of A is a real-valued matrix. $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix}.$$

Thus,

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4093 & -4092 \\ 3069 & -3068 \end{bmatrix}$$

Sample trajectories: outward lines parallel to v_2.

8. For the following vectors, determine if they are mutually orthogonal. Do they form a basis of $\mathbf{R^4}$? Compute unit vectors in the direction of each vector below.

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}.$$

Solution.

$u_1 \bullet u_1 = 18$,

$u_1 \bullet u_2 = 0$,

$u_1 \bullet u_3 = 0$,

$u_1 \bullet u_4 = 0$,

$u_2 \bullet u_2 = 36$,

$u_2 \bullet u_3 = 0$,

$u_2 \bullet u_4 = 0$,

$u_3 \bullet u_3 = 18$,

$u_3 \bullet u_4 = 0$,

$u_4 \bullet u_4 = 36$.

Thus, vectors are mutually orthogonal. They do form a basis because they are linearly independent. The corresponding unit vectors are

$$\begin{bmatrix} 0 \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \\ -1/3\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 1/2 \\ 5/6 \\ 1/6 \\ 1/6 \end{bmatrix}, \quad \begin{bmatrix} 1//3\sqrt{2} \\ 0 \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 5/6 \\ -1/2 \\ -1/6 \\ 1/6 \end{bmatrix}.$$

9. Find the angles (you may leave them in the form of arccos) formed by the following pairs of vectors:

a) $u = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

b) $u = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 4 \end{bmatrix}, v = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

c) $u = \begin{bmatrix} 17 \\ -17 \\ 17 \end{bmatrix}, v = \begin{bmatrix} -17 \\ 17 \\ 0 \end{bmatrix}$

Solution.

a). $\arccos(\frac{-16}{\sqrt{26}\sqrt{10}})$

b). $\arccos(\frac{-18}{\sqrt{43}\sqrt{14}})$

c). $\arccos(-\sqrt{2/3})$

10. Let $A = \begin{bmatrix} -\sqrt{3}/2 & 0.5 \\ -0.5 & -\sqrt{3}/2 \end{bmatrix}$.

- Describe geometrically the linear transformation $T : \mathbf{R^2} \to \mathbf{R^2}$ defined by the matrix A.
- Draw a typical trajectory of the discrete dynamical system defined by the matrix A.
- Describe geometrically the transformaion $T^{(10)}$ (the 10-th iteration of T).
- Write down the matrix A^{10}.

Solution.

Observe that $A = \begin{bmatrix} \cos(-5\pi/6) & -\sin(-5\pi/6) \\ \sin(-5\pi/6) & \cos(-5\pi/6) \end{bmatrix}$. Thus, T is a rotation by $\phi = -5\pi/6$. Therefore, $T^{(10)}$ is a rotation by $10 \bullet (-5\pi/6) = -25\pi/3 = -\pi/3$. The corresponding matrix $A^{10} = \begin{bmatrix} \cos(-\pi/3) & -\sin(-\pi/3) \\ \sin(-\pi/3) & \cos(-\pi/3) \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$.

Typical trajectory is a circle.

11. Give an example of a 2-dimensional discrete dynamical system with the following typical trajectory

a) Outward elliptical spiral :

$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} 2 \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$.

b) Inward circular spiral:

$A = 1/2 \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

c) Hyperbolic trajectories with (0,0) a repeller and the axes $(1,1)$ and $(-1,1)$:

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$