MATH 342 Lin Alg II Spring 2005
Practice problems for the Final exam **Name:** ____________________

1. Let $A = \begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.75 \end{bmatrix}$

   (a) Find eigenvalues and eigenvectors of A.

   (b) Consider a discrete dynamical system $x_{k+1} = Ax_k$. Classify the origin as an attractor, repellor or a saddle point of this dynamical system.

   Let x_0 be the initial state of the dynamical system defined above. Compute the state x_{100} of the system for

   1. $x_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$
   2. $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
   3. $x_0 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

   (d) What is the direction of the greatest repulsion and greatest attraction of the dynamical system above? Estimate the long term growth rate of x_k.

2. Let A be a square matrix of size $n \times n$ with *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ be the corresponding eigenvectors. Show that $v_1, v_2, \ldots v_n$ are linearly independent.

3. Let $T : \mathbf{R^3} \to \mathbf{R^3}$ be a linear transformation defined by the matrix $A = \begin{bmatrix} -1 & 0 & -3 \\ 3 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix}$, $T(x) = Ax$. Determine whether there exists a basis of $\mathbf{R^3}$ relative to which the matrix of T is diagonal.

4. Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$. Diagonalize A if possible.

5. a). Let $T : \mathbf{P^3} \to \mathbf{P^2}$ be a linear transformation given by the differential: for a polynomial $p(t)$,
$$T(p(t)) = p'(t).$$
   Compute the matrix of this linear transformation relative to the bases $< 1, t, t^2 >$ of $\mathbf{P^2}$ and $< 1, t, t^2, t^3 >$ of $\mathbf{P^3}$.

   b). Let $S : \mathbf{P^2} \to \mathbf{P^3}$ be a linear transformation given by the formula
$$S(p(t)) = \int_0^t p(s)ds$$
   Compute the matrix of S relative to the same bases as in (a).

c). **Without doing matrix multiplication**, answer the following question. Let A be the matrix of T (from (a)) and let B be the matrix of S (from (b)). Find AB.

6. Let $T : \mathbf{R^3} \to \mathbf{R^3}$ be a linear transformation defined by the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$.

   and let $\mathcal{B} =< b_1, b_2, b_3 >$ where $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Compute the matrix of T relative to the basis $\mathcal{B}$.

7. For each matrix below

   - Draw a few typical trajectories of the discrete dynamical system defined by the matrix A.

   - Compute A^{10}.

   (a) $A = \begin{bmatrix} -4 & -4 \\ 10 & 8 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$

8. Find the angles (you may leave them in the form of arccos) formed by the following pairs of vectors:

   a) $u = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

   b) $u = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 4 \end{bmatrix}, v = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

   c) $u = \begin{bmatrix} 17 \\ -17 \\ 17 \end{bmatrix}, v = \begin{bmatrix} -17 \\ 17 \\ 0 \end{bmatrix}$

9. Let $A = \begin{bmatrix} -\sqrt{3}/2 & 0.5 \\ -0.5 & -\sqrt{3}/2 \end{bmatrix}$.

   - Describe geometrically the linear transformation $T : \mathbf{R^2} \to \mathbf{R^2}$ defined by the matrix A.

   - Draw a typical trajectory of the discrete dynamical system defined by the matrix A.

   - Describe geometrically the transformaion $T^{(10)}$ (the 10-th iteration of T).

   - Write down the matrix A^{10}.

10. Give an example of a 2-dimensional discrete dynamical system with the following typical trajectory

    a) Outward elliptical spiral

    b) Inward circular spiral

    c) Hyperbolic trajectories with (0.0) a repeller and the axes $(1, 1,)$ and $(-1, 1)$.

11. Let $A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

    (a) Find eigenvalues of A.

    (b) Describe geometrically the linear transformation of the plane defined by the matrix A.

    (c) Describe geometrically the linear transformation defined by A^{12}. Compute A^{12}.

    (d) Draw a typical trajectory of the dynamical system defined by the matrix A. Is the origin an attractor or a repeller?

12. Let $T : \mathbf{R^2} \to \mathbf{R^2}$ be the linear transformation of the plane defined by the matrix $A = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}$, $T(x) = Ax$. Find a basis of $\mathbf{R^2}$ relative to which the matrix of T is diagonal.

13. Let $\mathbf{P_2}$ be the vector space of all polynomials of degree at most 2. Fix a basis $\mathcal{B} = \{1, t, t^2\}$ of $\mathbf{P_2}$.

    a) Let $T : \mathbf{P_2} \to \mathbf{P_2}$ be the linear transformation defined by the first derivative:

    $$T(p(t)) = p'(t).$$

    Compute the matrix A of the linear transformation T relative to the basis $\mathcal{B}$.

    b) Let $S : \mathbf{P_2} \to \mathbf{P_2}$ be the linear transformation defined by the second derivative:

    $$T(p(t)) = p''(t).$$

    Compute the matrix B of the linear transformation S relative to the basis $\mathcal{B}$.

    c) What is a numerical relationship between the matrices A and B?

14. Let

    $$A = \begin{bmatrix} 0.4 & 0.3 \\ -0.5 & 1.2 \end{bmatrix}$$

    be the matrix of the discrete dynamical system $x_{k+1} = Ax_k$ describing relative population of spotted owls and (thousands of) flying squirrels in the old-growth forest of Douglas fir.

    - Let $x_0 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ be the initial state of the discrete dynamical system defined by A . Write a general formula for the state x_k of the system.

- Classify the origin (point $(0,0)$) for this dynamical system (an attractor, a repeller or a saddle point).

- Draw several typical trajectories of the dynamical system defined by A.

- Show that both owls and squirrels will eventually perish. What should be the initial ratio between the numbers of owls and (thousands of) squirrels so that they perish the fastest? The slowest?

You may use $\chi_A(x) = x^2 - 1.6x + 0.63 = (x - 0.7)(x - 0.9)$.

15. Let u_1, u_2, u_3, u_4 be mutually orthogonal non-zero vectors. Show that they are linearly independent.

16. Let a, b be positive real numbers. Using Cauchy-Schwartz inequality, prove that

$$\sqrt{ab} \leq \frac{a+b}{2}$$

17. Let

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}.$$

Determine whether $< u_1, u_2, u_3, u_4 >$ form an orthogonal basis of $\mathbf{R^4}$. Is this an orthonormal basis?

Let

$$y = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$$

Compute coordinates of y relative to the basis $< u_1, u_2, u_3, u_4 >$.

18. Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$. Find basis and dimension of $(Col\, A)^{\perp}$.

19. Compute the QR-factorization of $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$.

20. Find least squares solutions and the least squares error for $Ax = b$ in two ways:

- computing the orthogonal projection of b onto $ColA$

- constructing the normal equations

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

21. Let $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

    a.) Compute A^{10}.

    b.) Compute a square root of A (i.e. a matrix B such that $B^2 = A$).

22. Find the best parabola $y(x) = ax^2 + bx + c$ which best fits the points $(1, 2)$, $(2, 2)$, $(1, 0)$ and $(3, 1)$.

23. Find the distance from the point $(1, 1, 1)$ to the plane $x - 2y + 3z = 0$.

24. a.) Define what does it mean for a matrix U to be orthogonal

    b.) Give an example of an orthogonal matrix U such that $U^T U = I$ but $UU^T \neq I$. Suggest a necessary and sufficient condition on an orthogonal matrix U to ensure that $UU^T = I$.

25. a.) Show that a 2×2 rotation matrix is necessarily orthogonal.

    b.) Show that any orthogonal 2×2 matrix with positive determinant is a rotation matrix.

26. Find a matrix A with the nullspace spanned by the vectors $(1, 2, 1, 2)$ and $(1, 1, 2, 2)$.

27. Let A be a symmetric 2×2 matrix.

    a.) Prove that eigenvalues of A are real.

    b.) Let λ_1, λ_2 be the eigenvalues, v_1, v_2 be corresponding eigenvectors, and assume $\lambda_1 \neq \lambda_2$. Show that v_1 is perpendicular to v_2.

28. Let $\mathbf{P}_3$ have the inner product given by evaluation at $(-2, -1, 1, 2)$. Find the best approximation of $t^3 + 17t$ by polynomials in the span of 1, t, $t^2 - 81t + 100$.

29. Let u, v be two unit vectors in an inner product space V. Show that $u + v$ is orthogonal to $u - v$.

30. Construct a symmetric matrix of size 3 with eigenvectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$.

31. The table below represents the test scores for a LA class.

	Quizzes	*MidtermI*	*MidtermII*	*Final*
Masha	80	85	75	180
Sasha	75	90	75	170
Petya	60	75	70	160
Fedya	90	90	85	180

Using Least Squares, find the linear function of the three other scores which gives the best approximation for the final score:

$$F \cong aQ + bM_1 + cM_2 + d$$

Using your approximation, predict the Final score for Jim who got 85 for Quizzes, 90 for Midterm I and 80 for Midterm II.

Note: Since these numbers are "realistic" feel free to use your calculator for this problem.

32. Find the distance from the point $(1, -7, 3)$ to the plane spanned by the vectors $(1, 1, 1)$ and $(1, 2, 3)$.

33. Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$.

(a) Find basis and dimension of $(Row\ A)^{\perp}$.

(b) Check whether b is orthogonal to the basis of $(Row\ A)^{\perp}$ that you found in (a). Make conclusion about the consistency of the system $Ax = b$ (is it consistent or not?)

(c) Find the least square solution and the least square error of the system $Ax = b$.

(d) Find a matrix B with the nullspace spanned by the vector b.

34. Let $\mathbf{P}_3$ have the inner product given by evaluation at $(-2, -1, 1, 2)$. Find the best approximation of the following functions by polynomials in the span of 1, t, $t^2 - 17t + 289$:

a.) t^2

b.) $t^3 - 3t$

35. For each set of eigenvalues and eigenvectors below, state whether there exists a symmetric matrix of size 2 with such eigenvalues and eigenvectors. If the answer is YES, give an example. If the answer is NO, justify it.

a.) $\lambda_1 = 1, v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$; $\lambda_2 = 2, v_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

b.) $\lambda_1 = 5, v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$; $\lambda_2 = -5, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

36. Use Cauchy-Schwarz inequality to show that

$$\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$$

37. Make a change of variable $x = Py$ which transform quadratic forms below into quadratic forms without cross-product terms. Give P and new quardatic forms. Classify quadratic forms. Sketch sample level curves.

    (a) $6xy + 8yz$.

    (b) $-5x^2 + 4xy - 2y^2$.

38. Find the maximal and minimal values of the function $f(x, y, z) = x + y + z$ on the unit sphere $x^2 + y^2 + z^2 = 1$.