MATH 342 Lin Alg II Spring 2005
Practice Midterm II **Name:** ______________________

1. Let a, b be positive real numbers. Using Cauchy-Schwartz inequality, prove that

$$\sqrt{ab} \leq \frac{a+b}{2}$$

2. Let

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}.$$

Determine whether $< u_1, u_2, u_3, u_4 >$ form an orthogonal basis of $\mathbf{R}^4$. Is this an orthonormal basis?

Let

$$y = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$$

Compute coordinates of y relative to the basis $< u_1, u_2, u_3, u_4 >$.

3. Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$. Find basis and dimension of $(Col\ A)^{\perp}$.

4. Compute the QR-factorization of $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$.

5. Find least squares solutions and the least squares error for $Ax = b$ in two ways:

   - computing the orthogonal projection of b onto $Col A$
   - constructing the normal equations

   (a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

   a.) Compute A^{10}.

   b.) Compute a square root of A (i.e. a matrix B such that $B^2 = A$).

7. Find the best parabola $y(x) = ax^2 + bx + c$ which best fits the points $(1, 2)$, $(2, 2)$, $(1, 0)$ and $(3, 1)$.

8. Find the distance from the point $(1, 1, 1)$ to the plane $x - 2y + 3z = 0$.

9. a.) Define what does it mean for a matrix U to be orthogonal

   b.) Give an example of an orthogonal matrix U such that $U^T U = I$ but $UU^T \neq I$. Suggest a necessary and sufficient condition on an orthogonal matrix U to ensure that $UU^T = I$.

10. a.) Show that a 2×2 rotation matrix is necessarily orthogonal.

    b.) Show that any orthogonal 2×2 matrix with positive determinant is a rotation matrix.

11. Find a matrix A with the nullspace spanned by the vectors $(1, 2, 1, 2)$ and $(1, 1, 2, 2)$.

12. Let A be a symmetric 2×2 matrix.

    a.) Prove that eigenvalues of A are real.

    b.) Let λ_1, λ_2 be the eigenvalues, v_1, v_2 be corresponding eigenvectors, and assume $\lambda_1 \neq \lambda_2$. Show that v_1 is perpendicular to v_2.

13. Let $\mathbf{P}_3$ have the inner product given by evaluation at $(-2, -1, 1, 2)$. Find the best approximation of $t^3 + 17t$ by polynomials in the span of 1, t, $t^2 - 81t + 100$.

14. Let u, v be two unit vectors in an inner product space V. Show that $u + v$ is orthogonal to $u - v$.

15. Construct a symmetric matrix of size 3 with eigenvectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$.

16. The table below represents the test scores for a LA class.

	Quizzes	*MidtermI*	*MidtermII*	*Final*
Masha	80	85	75	180
Sasha	75	90	75	170
Petya	60	75	70	160
Fedya	90	90	85	180

Using Least Squares, find the linear function of the three other scores which gives the best approximation for the final score:

$$F \cong aQ + bM_1 + cM_2$$

Using your approximation, predict the Final score for Jim who got 85 for Quizzes, 90 for Midterm I and 80 for Midterm II.

Note: Since these numbers are "realistic" feel free to use your calculator for this problem.