MATH 342 Lin Alg II Spring 2005
Practice Midterm **Name:** ____________________

1. Let $A = \begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.75 \end{bmatrix}$

   (a) Find eigenvalues and eigenvectors of A.

   (b) Consider a discrete dynamical system $x_{k+1} = Ax_k$. Classify the origin as an attractor, repellor or a saddle point of this dynamical system.

   Let x_0 be the initial state of the dynamical system defined above. Compute the state x_{100} of the system for

   1. $x_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$
   2. $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
   3. $x_0 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

   (d) What is the direction of the greatest repulsion and greatest attraction of the dynamical system above? Estimate the long term growth rate of x_k.

2. Let A be a square matrix of size $n \times n$ with *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ be the corresponding eigenvectors. Show that $v_1, v_2, \ldots v_n$ are linearly independent.

3. Let $T : \mathbf{R^3} \to \mathbf{R^3}$ be a linear transformation defined by the matrix $A = \begin{bmatrix} -1 & 0 & -3 \\ 3 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix}$, $T(x) = Ax$. Determine whether there exists a basis of $\mathbf{R^3}$ relative to which the matrix of T is diagonal.

4. Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$. Diagonalize A if possible.

5. a). Let $T : \mathbf{P^3} \to \mathbf{P^2}$ be a linear transformation given by the differential: for a polynomial $p(t)$,

   $$T(p(t)) = p'(t).$$

   Compute the matrix of this linear transformation relative to the bases $< 1, t, t^2 >$ of $\mathbf{P^2}$ and $< 1, t, t^2, t^3 >$ of $\mathbf{P^3}$.

   b). Let $S : \mathbf{P^2} \to \mathbf{P^3}$ be a linear transformation given by the formula

   $$S(p(t)) = \int_0^t p(s)ds$$

   Compute the matrix of S relative to the same bases as in (a).

c). **Without doing matrix multiplication**, answer the following question. Let A be the matrix of T (from (a)) and let B be the matrix of S (from (b)). Find AB.

6. Let $T : \mathbf{R^3} \to \mathbf{R^3}$ be a linear transformation defined by the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$.

   and let $\mathcal{B} =< b_1, b_2, b_3 >$ where $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Compute the matrix of T relative to the basis $\mathcal{B}$.

7. For each matrix below

   - Draw a few typical trajectories of the discrete dynamical system defined by the matrix A.

   - Compute A^{10}.

   (a) $A = \begin{bmatrix} -4 & -4 \\ 10 & 8 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$

8. For the following vectors, determine if they are mutually orthogonal. Do they form a basis of $\mathbf{R^4}$? Compute unit vectors in the direction of each vector below.

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}.$$

9. Find the angles (you may leave them in the form of arccos) formed by the following pairs of vectors:

   a) $u = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

   b) $u = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 4 \end{bmatrix}, v = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

   c) $u = \begin{bmatrix} 17 \\ -17 \\ 17 \end{bmatrix}, v = \begin{bmatrix} -17 \\ 17 \\ 0 \end{bmatrix}$

10. Let $A = \begin{bmatrix} -\sqrt{3}/2 & 0.5 \\ -0.5 & -\sqrt{3}/2 \end{bmatrix}$.

    - Describe geometrically the linear transformation $T : \mathbf{R^2} \to \mathbf{R^2}$ defined by the matrix A.

    - Draw a typical trajectory of the discrete dynamical system defined by the matrix A.

    - Describe geometrically the transformaion $T^{(10)}$ (the 10-th iteration of T).

    - Write down the matrix A^{10}.

11. Give an example of a 2-dimensional discrete dynamical system with the following typical trajectory

    a)

    b)

    c)