1. Let \( A = \begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.75 \end{bmatrix} \)

(a) Find eigenvalues and eigenvectors of \( A \).

(b) Consider a discrete dynamical system \( x_{k+1} = Ax_k \). Classify the origin as an attractor, repellor or a saddle point of this dynamical system.

Let \( x_0 \) be the initial state of the dynamical system defined above. Compute the state \( x_{100} \) of the system for

1. \( x_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \)
2. \( x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)
3. \( x_0 = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \)

(d) What is the direction of the greatest repulsion and greatest attraction of the dynamical system above? Estimate the long term growth rate of \( x_k \).

2. Let \( A \) be a square matrix of size \( n \times n \) with distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Let \( v_1, v_2, \ldots, v_n \) be the corresponding eigenvectors. Show that \( v_1, v_2, \ldots v_n \) are linearly independent.

3. Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a linear transformation defined by the matrix \( A = \begin{bmatrix} -1 & 0 & -3 \\ 3 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix} \),

\( T(x) = Ax \). Determine whether there exists a basis of \( \mathbb{R}^3 \) relative to which the matrix of \( T \) is diagonal.

4. Let \( A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix} \). Diagonalize \( A \) if possible.

5. a). Let \( T : \mathbb{P}^3 \rightarrow \mathbb{P}^2 \) be a linear transformation given by the differential: for a polynomial \( p(t) \),

\[ T(p(t)) = p'(t). \]

Compute the matrix of this linear transformation relative to the bases \( <1, t, t^2> \) of \( \mathbb{P}^2 \) and \( <1, t, t^2, t^3> \) of \( \mathbb{P}^3 \).

b). Let \( S : \mathbb{P}^2 \rightarrow \mathbb{P}^3 \) be a linear transformation given by the formula

\[ S(p(t)) = \int_0^t p(s) ds \]

Compute the matrix of \( S \) relative to the same bases as in (a).
c). **Without doing matrix multiplication**, answer the following question. Let $A$ be the matrix of $T$ (from (a)) and let $B$ be the matrix of $S$ (from (b)). Find $AB$.

6. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation defined by the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix}$.

   and let $B = b_1, b_2, b_3$ where $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Compute the matrix of $T$ relative to the basis $B$.

7. For each matrix below
   - Draw a few typical trajectories of the discrete dynamical system defined by the matrix $A$.
   - Compute $A^{10}$.

   (a) $A = \begin{bmatrix} -4 & -4 \\ 10 & 8 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$

8. For the following vectors, determine if they are mutually orthogonal. Do they form a basis of $\mathbb{R}^4$? Compute unit vectors in the direction of each vector below.

   $u_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}$, $u_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}$.

9. Find the angles (you may leave them in the form of $\arccos$) formed by the following pairs of vectors:

   a) $u = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

   b) $u = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 4 \end{bmatrix}$, $v = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

   c) $u = \begin{bmatrix} 17 \\ -17 \\ 17 \\ 17 \end{bmatrix}$, $v = \begin{bmatrix} -17 \\ 17 \\ 0 \end{bmatrix}$
10. Let \( A = \begin{bmatrix} -\sqrt{3}/2 & 0.5 \\ -0.5 & -\sqrt{3}/2 \end{bmatrix} \).

- Describe geometrically the linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by the matrix \( A \).
- Draw a typical trajectory of the discrete dynamical system defined by the matrix \( A \).
- Describe geometrically the transformation \( T^{(10)} \) (the 10-th iteration of \( T \)).
- Write down the matrix \( A^{10} \).

11. Give an example of a 2-dimensional discrete dynamical system with the following typical trajectory
   a)
   
   b)
   
   c)