MATH 256, Fall 2004
PRACTICE Midterm **Name:** ____________________

1. Solve initial value problems. Determine the longest interval where solution is well-defined. Describe the behaviour of the solution when $t \to \infty$ when applicable (converges, diverges, bounded, unbounded)

   (a) $y' = t, \quad y(2) = 5$ Describe behaviour at ∞.

   Answer. $y(t) = t^2/2 + 3$, diverges.

   (b) $y'' + e^t y' + t^3 y = 0, \quad y(1) = 0, y'(1) = 0$

   Answer. $y(t) = 0$

   (c) $ty' + 2y = \sin t, \quad y(1) = \sin 1 - \cos 1 + 1$

   Describe behaviour at ∞.

   Answer. $y(t) = \frac{\sin t}{t^2} - \frac{\cos t}{t} + \frac{1}{t^2}$, converges to 0 at $+\infty$.

   (d)$y' = 2t/(y + t^2 y), \quad y(0) = -2$

   Answer. $y(t) = -\sqrt{2\ln(1+t^2) + 4}$

   (e) $y' = 2y - 5y^3, \quad y(0) = 1$ (Bernoulli equation)

   Answer. $y(t) = \frac{1}{\sqrt{2.5 - 1.5e^{-4t}}}$

2. Find a fundamental set of solutions, compute Wronskian, give a general solution and then solve the initial value problem, sketch the graph of the solution and describe behaviour at ∞.

   (a)$y'' + 5y' + 6y = 0, \quad y(0) = 2, y'(0) = 2$

   Answer.
   $< e^{-2t}, e^{-3t} >$,
   $W = -e^{5t}$,
   $y(t) = 8e^{-2t} - 6e^{-3t}$,
   converges to 0 at $+\infty$, diverges to $-\infty$ at $-\infty$.

   (b)$y'' + 2y' + 6y = 0, \quad y(0) = 2, y'(0) = 2$

   Answer.
   $< e^{-t}\cos(\sqrt{5}t), e^{-t}\sin(\sqrt{5}t) >$,
   $W = e^{2t}$,
   $y(t) = 2e^{-t}\cos(\sqrt{5}t) + 4e^{-t}\sin(\sqrt{5}t)$,
   converges to 0 at $+\infty$, diverges at $-\infty$.

3. Compute Wronskian of $\arcsin(x)$ and $\arccos(\sqrt{1-x^2})$. Make conclusion about linear dependence of these two functions.

   You may use $\arcsin' x = 1/\sqrt{1-x^2}, \arccos' x = -1/\sqrt{1-x^2}$.

Answer. These functions are the same! Hint: draw a right triangle with sides x and $\sqrt{1-x^2}$.

Wronskian is 0. Functions are linearly dependent.

4. Someone is buying a house anticipating to be able to pay $1200(1+t/60)$ where t is the number of months since the loan was made. Assuming the interest rate to be 5%, compute the price the person can afford if he wishes to pay off the loan in 20 years.

Answer.

Let's measure t in years. Then the payment rate will become $1200(1+t/5)$. Let $r = 0.05$. Let $S(t)$ be the amount owed at time t.

Equation: $S'(t) = rS(t) - 1200(1+t/5)$ (= interest added - payment made).

Rewrite as $S'(t) - rS(t) = -1200 - 240t$. This is a linear equation. To solve, multiply by the integration factor e^{-rt} and integrate

$$e^{-rt}S(t) = -\int(1200e^{-rt} + 240te^{-rt})dt = 1200\frac{e^{-rt}}{r} + 240t\frac{e^{-rt}}{r} + 240\frac{e^{-rt}}{r^2} + c$$

Divide by e^{-rt}.

$$S(t) = \frac{1}{r}(1200 + 240t + 240/r) + ce^{rt}$$

Plug in $r = 0.05 = 1/20$

$$S(t) = 20(6000 + 240t) + ce^{t/20}$$

It is given that $S(20) = 0$. Plug in $t = 20$

$$0 = 20(6000 + 4800) + c * e$$

Thus,

$$c = -216000/e$$

We get

$$S(t) = 20(6000 + 240t) - 216000e^{t/20-1}$$

Finally, by plugging in $t = 0$, we get the price that the person can afford to pay:

$$S(0) = 20 * 6000 - 216000/e \simeq 160000$$

5. For the following equations, draw the phase line, determine equilibrium solutions, classify them as asymptotically stable, unstable or semistable. Sketch several graphs of solutions on the ty-plane. Find general solution of the equation.

(a) $y' = -17(y-2)^2$

Answer. $y = 2$, semistable. General solution $y(t) = 1/(17t + c) + 2$

(b) $y' = (y-1)(y-3)$

Answer. $y = 1$, stable, $y = 3$, unstable. General: $y(t) = \frac{3-ce^{2t}}{1-ce^{2t}}$

6. Approximate the solution of the initial value problem $y' = 2y - 1, \quad y(1) = 0$ at $t = 2$. Take $h = 0.25$. Solve the equation, compare your approximation with the exact value of the solution at $t = 2$ and estimate the error.

   Answer. Euler's method yeilds $y_4 \simeq -2.03$.

   Solution: $y(t) = \frac{1-e^{2t-2}}{2}$. $y(2) = -3.19$. Error is about 30%.

7. Is it possible that the (missing) picture represents integral curves for the equation

   (a) $y' = y^{1/3}$

   Answer. Yes. $\delta f/\delta y = 1/3y^{-2/3}$ - discontinuous at $y = 0$. thus, the Uniqueness theorem does not apply and one can have intersecting integral curves.

   (b) $y' = y^{4/3}$

   Answer. No. Here $\delta f/\delta y = 4/3y^{1/3}$ is continuous. Thus, the uniqueness theorem implies that there is only one integral curve at any point: no intersections.

   (c) $t^2 y'' = y$

   Answer. Yes. In the equation $y'' - 1/t^2 y = 0$ the coefficient of y is undefined at $t = 0$. Thus, the uniqueness theorem does not apply.

   (d) $y'' = t^2 y$

   Answer. Yes. Even though uniqueness theorem applies here (coefficients are continuous everywhere), the curves on the picture have different slopes at their intersection point. Thus, they can be solutions of different initial value problems involving this equation. (For an initial value problem of a second order equation we have to specify both $y(t_0)$ and $y'(t_0)$).

8. State the longest interval where the solution of the initial value problem

$$(t+1)y'' + \frac{1}{t-1}y' + (\ln t)\, y = 0, \quad y(1/2) = 0, y'(1/2) = 1$$

   exists. Is solution unique? Explain.

   Answer. (0.1). The solution is unique because this is a linear equation with coefficients continuous on the interval $(0, 1)$ containing the initial point $t = 1/2$

9. Does the initial value problem

$$y' = t^2 + y^2, \quad y(1) = 2, y'(1) = 239$$

   have unique solution?

   Answer. Yes. Both $f(t, y) = t^2 + y^2$ and $\delta f/\delta y = 2y$ are continuous functions, therefore, uniqueness theorem applies.