MATH 256, Fall 2004
Midterm **Name:** ____________________

(20) 1. Find general solutions of the ODE

(a) $y' - 2y = e^{3t}$

Answer. $e^{3t} + ce^{2t}$

(b) $y' = y^2/t$

Answer. $-\frac{1}{\ln|t|+c}$

(20) 2. Solve the initial value problem. Predict long term behaviour of the solution (behaviour at ∞).

$y'' - 5y' + 6y = 0, \quad y(0) = 3, y'(0) = 7.$

Answer. $e^{3t} + 2e^{2t}$, diverges at ∞.

3. Check all terms that apply to the equation

$$y'' - 2ty = 0$$

1. differential equation
2. first order
3. second order
4. linear
5. non-linear
6. autonomous
7. separable
8. homogeneous
9. non-homogeneous

(For each marked item you get +1pt if it applies and -1pt if it does not)

Answer.

1. differential equation

3. second order

4. linear

7. separable

8. homogeneous

(8) 4. (a) Give an example of a second order linear non-homogeneous equation with non-constant coefficients.

Answer. For example, $y'' - 2ty = t$

(b) Does the following apply to your example above : " if $y_1(t), y_2(t)$ are solutions of the equation then $ay_1(t) + by_2(t)$ is a solution for any choice of numbers a, b"?

Answer. No, because it is NON-homogeneous.

(20) 5. $y' = y^2 - 4$.

(a) Find equilibrium solutions

Answer. $y = 2, y = -2$

(b) Draw the phase line

Answer. $\rightarrow -2 \leftarrow 2 \rightarrow$

(c) Classify equilibrium solutions as stable, unstable or semistable

Answer. $y = -2$ is asymptotically stable, $y = 2$ is unstable

(d) Sketch several solutions on the ty plane. Using your sketch, predict long time behaviour (behaviour at ∞) of the solution of the initial value problem $y(0) = -3/2$.

Answer. Converges to -2.

(20) 6. It is being observed that the number of poorly designed ballots at the presidential elections that later have to be disqualified grows by 20% with every election (or 5% yearly if one assumes that ballots are made continuously between the elections). Assuming that the number of registered voters, and, thus, the total number of ballots, remains constant over the years and that the number of faulty ballots in 2000 consituted 5% of all ballots, predict in what year the elections will come to a halt because of no qualified ballots to count. You may use $\ln 20 \simeq 3$.

Answer. Let $F(t)$ be the function which counts the number of badly designed ballots at the time t. Then,

$$F'(t) = 0.05F(t)$$

Solving for F, we get

$$F(t) = F_0 e^{0.05t},$$

where F_0 is the number of bad ballots in 2000 which is when we set up our timer. Let R be the total number of ballots. Then $F_0 = 0.05R$. Thus,

$$F(t) = 0.05Re^{0.05t} = \frac{Re^{t/20}}{20}$$

We would like to know when $F(t)$ becomes R. Thus, we have to solve for t:

$$\frac{Re^{t/20}}{20} = R$$

Solving for t, we get $t = 20\ln(20) \simeq 60$. Thus, the answer is 2060.

(7) 7. Explain why the picture below cannot represent integral curves of the equation $y' + e^{t^2} y = 0$

Sorry - no picture.

Answer. This is a 1st order linear equation with continuous coefficients. Thus, the solution for any inital condition $y(t_0) = y_0$ is unique by the Existence and Uniqueness theorem. Therefore, integral curves cannot intersect (this would contradict uniqueness of the solution).

(10) 8. (Bonus.) Consider an equation $y'' + p(t)y' + q(t) = 0$, where $p(t), q(t)$ are continuous functions. Let $(y_1(t), y_2(t))$ be a fundamental set of solutions, and let $(u_1(t), u_2(t))$ b another fundamental set of solutions. Compute the first derivative of the function $\frac{W(y_1,y_2)}{W(u_1,u_2)}$

Answer. By Abel's theorem, $W(y_1, y_2) = c_1 e^{\int p(t)dt}$, where c_1 is some non-zero number. Similarly, $W(u_1, u_2) = c_2 e^{\int p(t)dt}$. Thus, $\frac{W(y_1,y_2)}{W(u_1,u_2)} = c_1/c_2$, a constant function. Derivative of a constant function is always 0. Thus, the answer is 0.