MATH 256, Fall 2004
FINAL **Name:** ____________________

No notes, books or calculators. SHOW ALL YOUR WORK

(10) 1. Solve the initial value problem

$y' = t, quady(0) = 0$

Answer. $y = t^2/2.$

(20) 2. $y' + 2y = e^{-t}$

Answer. $y = e^{-t} + ce^{-2t}$

(20) 3. $y' = 2t/(y + t^2 y)$

Answer. $y^2 = 2\ln(t^2 + 1) + c$

(20) 4. $y^{(4)} - 8y' = 0$

Answer. $c_1 + c_2 e^{2t} + c_3 e^{-t}\cos(\sqrt{3}t) + c_4 e^{-t}\sin(\sqrt{3}t)$

(25) 5. Consider the differential equation $y'' + 5y' + 6y = 0.$

(10) (a) Find fundamental set of solutions

Answer. $< e^{-3t}, e^{-2t} >$

(10) (b) Compute Wronskian

Answer. $W = e^{-5t}$

(5) (c) Are the following pairs of functions linearly independent? Give a short explanation

(c.1) $< e^{-3t}, e^{-2t} >$

Answer. Yes. $W(e^{-3t}, e^{-2t}) \neq 0$, thus, they are linearly independent.

(c.2) $< -3e^t, -2e^t >$

Answer. No. We have $-2e^t = (\frac{2}{3})(-2e^t)$. Thus, functions are linearly dependent.

(20) 6. Find the general solution of

$$y'' - 5y' + 6y = e^{2t}$$

Answer. $y = c_1 e^{3t} + c_2 e^{2t} - te^{2t}$

(30) 7. Consider the differential equation $y' = 2y - y^2$.

(1) (a) Is this an autonomous equation? **Answer.** Yes.

(1) (b) Is this a separable equation? **Answer.** Yes.

(1) (c) Is this a linear equation? **Answer.** No.

(3) (d) Find equilibrium solutions.

Answer. $y(t) = 0, y(t) = 2$

(7) (e) Draw the phase line and sketch several integral curves.

(3) (f) Classify equilibrium solutions as asympotically stable, semistable or unstable.

Answer. $y(t) = 0$ is unstable, $y(t) = 2$ is asymptotically stable.

(10) (g) Consider the initial value problem $y' = 2y - y^2$, $y(0) = 1$ (still the same equation). Using Euler's method with the step $h = 0.5$, approximate the solution at the point $t = 1$.

Answer. Let $f(y) = 2y - y^2$.

$y_0 = 1$

$y_1 = y_0 + f(y_0)0.5 = 1.5$

$y_2 = y_1 + f(y_1)0.5 = 1.5 + (2 \bullet 1.5 - 1.5^2) = 1.5 + 0.75 \bullet 0.5 = \mathbf{1.875}$

(alternatively, in case you prefer fractions, $y_2 = 3/2 + (2 \bullet 3/2 - (3/2)^2) \bullet 1/2 = 3/2 + (3 - 9/4) \bullet 1/2 = 3/2 + 3/4 \bullet 1/2 = 3/2 + 3/8 = \mathbf{15/8} = \mathbf{1\frac{7}{8}}$)

(2) (h) Describe long term behaviour of the solution of the initial value problem above.

Answer. $y(t) \to 2$

(2) (i) For a small step h, would you expect the Euler's method to produce good approximation for this particular initial value problem? Explain.

Answer. Yes, because $y(t)$ converges to 2

(30) 8. Find a general solution of the ODE:

$$y'' + y = \frac{1}{\sin t}$$

Answer. $c_1 \cos t + c_2 \sin t - t\cos t - \sin t \ln|\sin t|$

(15) 9. Give an example of a differential equation which has solutions e^t, te^t.

Answer. $y'' - 2y' + y = 0$

(20) 10. Find the first 5 terms of the power series solution of the equation $y'' - 2xy = 0$ at $x = 0$.

Answer. $y(x) = a_0 + a_1 x + a_0 \frac{x^3}{3} + a_1 \frac{x^4}{6} + \ldots$