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Math 583 Spring 2014 Group Cohomology
Homework 1
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Problem 1

(1) Let $E^2_{pq} \Rightarrow H_{p+q}$ be a first quadrant (homological) spectral sequence converging to H_*. Show that there is an exact sequence ("The five-term exact sequence"):

$$H_2 \to E^2_{20} \xrightarrow{d^2} E^2_{01} \to H_1 \to E^2_{10} \to 0.$$

(2) Formulate and prove an analogous statement for a first quadrant cohomological spectral sequence.

For (1), let $\delta = d^2_{20} \colon E^2_{20} \to E^2_{01}$. It's easy to see that if we take homology at the second page, the third page is

$$\begin{matrix} E^2_{01}/\operatorname{im}\delta & \cdot & \cdot \\ \\ E^2_{00} & E^2_{10} & \ker\delta \end{matrix}$$

whereupon the four indicated terms stabilize. Since this is a first quadrant spectral sequence, it's also easy to see the relationship between these E^∞_{pq} terms and the filtration on H_{p+q} is

$$\begin{matrix} E^\infty_{01} = F_0H_1 & \cdot & \cdot \\ \\ E^\infty_{00} = H_0 & E^\infty_{10} = H_1/F_0H_1 & E^\infty_{20} = H_2/F_1H_2 \end{matrix}$$

Hence we have a sequence

$$H_2 \twoheadrightarrow H_2/F_1H_2 \cong \ker\delta \hookrightarrow E^2_{20} \xrightarrow{\delta} E^2_{01} \twoheadrightarrow E^2_{01}/\operatorname{im}\delta \cong F_0H_1 \hookrightarrow H_1 \twoheadrightarrow H_1/F_0H_1 \cong E^2_{10} \to 0$$

One can quickly check exactness of the induced five-term sequence, so the result follows.

For (2), let $\delta = d^{01}_2 \colon E^{01}_2 \to E^{20}_2$. The third page is

$$\begin{matrix} \ker\delta & \cdot & \cdot \\ \\ E^{00}_2 & E^{10}_2 & E^{20}_2/\operatorname{im}\delta \end{matrix}$$

which again collapses, and the E^{pq}_∞ terms relate to the filtration on H^* via

$$\begin{matrix} E^{01}_\infty = H^1/F^1H^1 & \cdot & \cdot \\ \\ E^{00}_\infty = H^0 & E^{10}_\infty = F^1H^1 & E^{20}_\infty = F^2H^2 \end{matrix}$$

Hence we have a sequence

$$0 \to E_2^{10} \cong F^1H^1 \hookrightarrow H^1 \twoheadrightarrow H^1/F^1H^1 \cong \ker \delta \hookrightarrow E_2^{01} \xrightarrow{\delta} E_2^{20} \twoheadrightarrow E_2^{20}/\operatorname{im} \delta \cong F^2H^2 \hookrightarrow H^2$$

which induces the exact sequence

$$0 \to E_2^{10} \to H^1 \to E_2^{01} \to E_2^{20} \to H^2.$$

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Problem 2 Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of complexes. Using spectral sequences, show that there is an exact sequence in homology:

$$\cdots \to H_{n+1}(C_*) \to H_n(A_*) \to H_n(B_*) \to H_n(C_*) \to H_{n-1}(A_*) \to \cdots$$

Consider the double complex whose bottom two rows are

$$\begin{array}{ccccccccc}
0 & \longleftarrow & C_1 & \longleftarrow & B_1 & \longleftarrow & A_1 & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & C_0 & \longleftarrow & B_0 & \longleftarrow & A_0 & \longleftarrow & 0
\end{array}$$

(where as usual we've toggled the sign on the B_* column's maps). Taking horizontal homology gives 0's everywhere since the rows are exact, so ${}^{II}E^r_{pq}$ collapses to 0 at $r = 1$, forcing the abutment to be trivial. Hence ${}^{I}E^r_{pq} \Rightarrow 0$. Take vertical homology to get ${}^{I}E^1_{pq}$ as

$$\begin{array}{ccccccccc}
0 & \longleftarrow & H_1(C_*) & \xleftarrow{\beta_1} & H_1(B_*) & \xleftarrow{\alpha_1} & H_1(A_*) & \longleftarrow & 0 \\
 & & & & & & & & \\
0 & \longleftarrow & H_0(C_*) & \xleftarrow{\beta_0} & H_0(B_*) & \xleftarrow{\alpha_0} & H_0(A_*) & \longleftarrow & 0
\end{array}$$

It's easy to see the B_* column stabilizes on the next page, so it must stabilize at 0, i.e. the above is exact at $H_n(B_*)$. Take horizontal homology to get ${}^{I}E^2_{pq}$ as

$$\begin{array}{ccc}
H_1(C_*)/\operatorname{im}\beta_1 & 0 & \ker \alpha_1 \\
 & \nwarrow & \\
H_0(C_*)/\operatorname{im}\beta_0 & 0 & \ker \alpha_0
\end{array}$$

It's easy to see the C_* and A_* columns stabilize on the next page, so the above is exact at $\ker \alpha_0$ and $H_1(C_*)/\operatorname{im}\beta_1$, i.e. the connecting map is an isomorphism. Hence we have a sequence

$$\cdots \to H_1(A_*) \xrightarrow{\alpha_1} H_1(B_*) \xrightarrow{\beta_1} H_1(C_*) \twoheadrightarrow H_1(C_*)/\operatorname{im}\beta_1 \cong \ker \alpha_0 \hookrightarrow H_0(A_*) \xrightarrow{\alpha_0} \cdots$$

which gives the desired long exact sequence. ■

Problem 3 Prove a subtler version of the **5-lemma**: namely, what are the "minimal" conditions you need to put on the following commutative diagram with exact rows to conclude that γ is injective? What about surjective?

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Consider the diagram as a double complex by flipping it horizontally and toggling the signs of the second and fourth vertical arrows without loss of generality

We find ${}^{II}E^1_{pq}$ is

$$\begin{array}{ccccc} \cdot & 0 & 0 & 0 & \cdot \\ \downarrow & & & & \downarrow \\ \cdot & 0 & 0 & 0 & \cdot \end{array}$$

where $\cdot$ represents the kernel or cokernel of the appropriate map. The remaining pages do not change the $n = 2$ and $n = 3$ antidiagonals, hence the filtration on these pieces of the abutment H_n are trivial, so $H_2 = H_3 = 0$. In particular ${}^{I}E^{\infty}_{pq} = 0$ for $p + q = n = 2, 3$.

Now compute ${}^{I}E^1_{pq}$:

$$\ker \epsilon \longleftarrow \ker \delta \longleftarrow \ker \gamma \longleftarrow \ker \beta \longleftarrow \ker \alpha$$

$$\operatorname{coker} \epsilon \longleftarrow \operatorname{coker} \delta \longleftarrow \operatorname{coker} \gamma \longleftarrow \operatorname{coker} \beta \longleftarrow \operatorname{coker} \alpha$$

If $\ker \delta = \ker \beta = \operatorname{coker} \alpha = 0$, taking homology at $\ker \gamma$ does nothing at this page or the next, so $\ker \gamma = {}^{I}E^{\infty}_{21} = 0$. Likewise if $\operatorname{coker} \delta = \operatorname{coker} \beta = \ker \epsilon$, it follows that $\operatorname{coker} \gamma = 0$. So, we have

- γ is injective if δ, β are injective and α is surjective
- γ is surjective if δ, β are surjective and ϵ is injective

This seems to essentially be the two "four lemmas"; I'm not sure if this is a "minimal" set of conditions in any reasonable sense. They seem to be the most obvious conditions, if that's worth anything. ∎

Problem 4 Let $f\colon (A_*, d_A) \to (B_*, d_B)$ be a map of complexes. The mapping cone $\operatorname{Cone}(f)_*$ is the total complex of the double complex $A_* \xrightarrow{f} B_*$. It can be described explicitly as follows:

$$\operatorname{Cone}(f)_n := A_{n-1} \oplus B_n, \qquad d_n\colon A_{n-1} \oplus B_n \xrightarrow{\begin{pmatrix} -d_A & 0 \\ -f & d_B \end{pmatrix}} A_{n-2} \oplus B_{n-1}.$$

Show that there is a long exact sequence

$$\cdots \to H_{n+1}(\operatorname{Cone}(f)_*) \to H_n(A_*) \to H_n(B_*) \to H_n(\operatorname{Cone}(f)_*) \to H_{n-1}(A_*) \to \cdots.$$

Minor note: I assume the double complex $A_* \xrightarrow{f} B_*$ is anticommutative, whereas a "map of complexes" in my experience has commutative squares.

Consider the double complex

$$\begin{array}{ccccccccc}
0 & \longleftarrow & A_{n-1} & \overset{\pi_A}{\twoheadleftarrow} & A_{n-1} \oplus B_n & \hookleftarrow & B_n & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle -d_A} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_B} & & \\
0 & \longleftarrow & A_{n-2} & \overset{\pi_A}{\twoheadleftarrow} & A_{n-2} \oplus B_{n-1} & \hookleftarrow & B_{n-1} & \longleftarrow & 0
\end{array}$$

One can check this has exact rows and columns. Hence we have an exact sequence of chain complexes

$$0 \to B_* \to \mathrm{Cone}(f)_* \to A[-1]_* \to 0,$$

which from Problem 2 gives rise to a long exact sequence

$$\cdots \to H_{n+1}(\mathrm{Cone}(f)_*) \to H_n A_* \to H_n(B_*) \to H_n(\mathrm{Cone}(f)_*) \to H_{n-1}(A_*) \to \cdots .$$

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Problem 5 Establish the Künneth spectral sequence for complexes (it's ok to use the classical Künneth formula as in [Weibel, 3.6.3] if you feel that you need to): Let R be a (commutative) ring and C_*, D_* complexes of R-modules bounded below. Assume the C_n are flat for all n. Show that there is a convergent spectral sequence

$$E^2_{pq} = \bigoplus_{s+t=q} \mathrm{Tor}^R_p(H_s(C_*), H_t(D_*)) \Rightarrow H_{p+q}(C_* \otimes_R D_*)$$

where $H_{p+q}(C_* \otimes_R D_*)$ stands for the homology of the total complex.

Let $P_{*\cdot} \to D_*$ be a Cartan-Eilenberg resolution. Consider the double chain complex

$$E^0_{pq} = \mathrm{Tot}_q(C_* \otimes P_{*p})$$

where the vertical maps are the usual total complex maps, and the horizontal maps are induced by the horizontal maps from $P_{*\cdot} \to D_*$. Take horizontal homology to get

$$\begin{aligned}
({}^{II}E^1_{pq})^T &= H_p(\mathrm{Tot}_q(C_* \otimes_R P_{*\cdot})) = H_p\left(\bigoplus_{s+t=q} C_s \otimes_R P_{t\cdot}\right) \\
&= \bigoplus_{s+t=q} H_p(C_s \otimes_R P_{t\cdot}) = \bigoplus_{s+t=q} C_s \otimes_R H_p(P_{t\cdot}) = \bigoplus_{s+t=q} C_s \otimes_R \delta_{p0} D_t \\
&= \delta_{p0}\, \mathrm{Tot}_q(C_* \otimes_R D_*),
\end{aligned}$$

where we've used the following facts: $\oplus$ is finite; homology commutes with finite sums; homology commutes with $C_s \otimes_R -$ since C_s is flat; $P_{t\cdot} \to D_t$ is a projective resolution, so taking homology gives zeros except at the very bottom when it gives D_t. That is, we're left with just $\mathrm{Tot}_q(C_* \otimes_R D_*)$ in the $p = 0$ column. Thus we'll be able to recover the abutment exactly, not just the associated graded object. Take vertical homology to get

$$({}^{II}E^2_{pq})^T = \delta_{p0} H_q(\mathrm{Tot}_\circ(C_* \otimes_R D_*)).$$

The sequence stabilizes here, so the nth piece of the abutment is $H_n(\mathrm{Tot}_\circ(C_* \otimes_R D_*))$ (since $p = 0$ gives the only non-zero term and $n = p + q$).

On the other hand, take vertical homology of E^0_{pq} to get

$$^{I}E^1_{pq} = H_q(\mathrm{Tot}_\circ(C_* \otimes_R P_{*p}))$$

and then take horizontal homology to get

$$\begin{aligned} ^{I}E^2_{pq} &= H_p(H_q(\mathrm{Tot}_\circ(C_* \otimes_R P_{*\cdot}))) = H_p\left(\bigoplus_{s+t=q} H_s(C_*) \otimes_R H_t(P_{*\cdot})\right) \\ &= \bigoplus_{s+t=q} H_p(H_s(C_*) \otimes_R H_t(P_{*\cdot})) = \bigoplus_{s+t=q} \mathrm{Tor}^R_p(H_s(C_*), H_t(D_*)), \end{aligned}$$

where we've used the following facts: the Künneth formula quoted below; $\oplus$ is finite; homology commutes with finite sums; $H_t(P_{*\cdot}) \to H_t(D_*)$ is a projective resolution; Tor is computed in the usual way. This application of the Künneth formula uses the fact that $P_{*\cdot}$ is projective, hence flat, and $B(P_{*\cdot})$ is projective, hence flat, which was proved in class; it also uses the fact that $H_t(P_{*\cdot})$ is projective, hence flat, so the Tor term vanishes, and we have an isomorphism. In all we have a (convergent) spectral sequence

$$\bigoplus_{s+t=q} \mathrm{Tor}^R_p(H_s(C_*), H_t(D_*)) \Rightarrow H_{p+q}(C_* \otimes_R D_*).$$

1 Theorem (Künneth formula for complexes)

Let P and Q be right and left R-module complexes, respectively. If P_n and $d(P_n)$ are flat for each n, then there is an exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes H_q(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{p+q=n-1} \mathrm{Tor}^R_1(H_p(P), H_q(Q)) \to 0.$$

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