**Homework 2 for 583, Spring 2014**

due Friday, May 30, 2014

**Problem 1.** Prove "Tensor identity": Let $H < G$ be groups, let $M$ be an $H$-module and $N$ a finite dimensional $G$-module. Then there is a natural isomorphism

$$\operatorname{Coind}_H^G(M \otimes N{\downarrow_H}) \simeq \operatorname{Coind}_H^G M \otimes N$$

**Problem 2.** Let $H \subset G$ be an embedding of groups. Prove that the following diagram commutes:

$$\begin{array}{ccc} H^*(G,M) & \xrightarrow{\ \operatorname{res}^*\ } & H^*(H,M) \\ & \searrow^{\simeq} & \downarrow \\ & & H^*(H, \operatorname{Coind}_H^G M) \end{array}$$

where the top map is the restriction map in cohomology, the diagonal map is the Frobenius isomorphism, and the vertical map is induced by the canonical map $M \to \operatorname{Coind}_H^G M$.

**Problem 3.** Prove "double coset formula":

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G M = \bigoplus_{x \in K\backslash G/H} \operatorname{Ind}_{K\cap xHx^{-1}}^K \operatorname{Res}_{K\cap xHx^{-1}}^{xHx^{-1}} xM$$

where $K, H \subset G$ are subgroups of finite index, $M$ is a $G$-module and $K\backslash G/H$ is a set of double coset representatives.

**Problem 4.** Let $m$ be an odd integer, and let $D_{2m} = C_m \rtimes C_2$ be the dihedral group. Compute $H^*(D_{2m}, \mathbb{Z})$ as $\mathbb{Z}$-algebra.

*Hint.* You will need to compute the action of $C_2$ on $H^q(C_m, \mathbb{Z})$. You can use the explicit map on periodic resolutions constructed in Example 6.7.10 in Weibel to compute this action.

**Problem 5.** "Dimension shifting". Let $G$ be a finite group, $M$ b a $G$-module and $P. \to M$ be a minimal projective resolution of $M$. The $i$th syzygy (or Heller shift) of $M$ is defined as follows:

$$\Omega M := \ker\{P_0 \to M\}, \quad \Omega^n M := \ker\{P_{n-1} \to P_{n-2}\} \text{ for } n \geq 2.$$

Let $N$ be a simple $G$-module. Show that there is an isomorphism

$$\operatorname{Ext}_G^i(M,N) \cong \operatorname{Hom}_G(\Omega^i M, N)$$

for any $i > 0$.

## References

[1] C. Weibel, An introduction to homological algebra, Cambridge University Press, 1995