Homework 1 for 583, Spring 2014

due Friday, May 2, 2014

Throughout, we assume that we live in some nice abelian category.

Problem 1.

(1) Let $E^2_{pq} \Rightarrow H_{p+q}$ be a first quadrant (homologial) spectral sequence converging to H_*. Show that there is an exact sequence ("The five-term exact sequence"):

$$H_2 \longrightarrow E^2_{20} \xrightarrow{d^2} E^2_{01} \longrightarrow H_1 \longrightarrow E^2_{1[} \longrightarrow 0$$

(2) Formulate and prove an analogous statement for a first quadrant cohomological spectral sequence.

Problem 2. Let $0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$ be a short exact sequence of complexes. Using spectral sequences, show that there is an exact sequence in homology:

$$\ldots \longrightarrow H_{n+1}(C_*) \longrightarrow H_n(A_*) \longrightarrow H_n(B_*) \longrightarrow H_n(C_*) \longrightarrow H_{n-1}(A_*) \longrightarrow \ldots$$

Problem 3. Prove a subtler version of the **5-lemma**: namely, what are the "minimal" conditions you need to put on the following commutative diagram with exact rows to conclude that γ is injective? What about surjective?

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Problem 4. Let $f : (A_*, d_A) \to (B_*, d_B)$ be a map of complexes. The mapping cone $\mathsf{Cone}(f)_*$ is the total comlex of the double complex $A_* \xrightarrow{f} B_*$. It can be described explicitly as follows:

$$\mathsf{Cone}(f)_n = A_{n-1} \oplus B_n, \quad d_n : A_{n-1} \oplus B_n \xrightarrow{\begin{pmatrix} -d_A & 0 \\ -f & d_B \end{pmatrix}} A_{n-2} \oplus B_{n-1}.$$

Show that there is a long exact sequence

$$\ldots \longrightarrow H_{n+1}(C_*) \longrightarrow H_n(A_*) \longrightarrow H_n(B_*) \longrightarrow H_n(\mathsf{Cone}(f)_*) \longrightarrow H_{n-1}(A_*) \longrightarrow \ldots$$

Problem 5. Establish the Künneth spectral sequence for complexes (it's ok to use the classical Künneth formula as in [1, 3.6.3] if you feel that you need to):
Let R be a (commutative) ring, and C_*, D_* be complexes of R-modules bounded below. Assume that C_n are flat for all n. Show that there is a convergent spectral sequence

$$E^2_{pq} = \bigoplus_{s+t=q} \mathrm{Tor}^R_p(H_s(C_*), H_t(D_*)) \Rightarrow H_{p+q}(C_* \otimes_R D_*)$$

where $H_{p+q}(C_* \otimes_R D_*)$ stands for the homology of the total complex.

References

[1] C. Weibel, An introduction to homological algebra, Cambridge University Press, 1995