Sample MIDTERM II, version 1
Last year’s midterm for Spring MATH 126 A, B

Scientific, but not graphing calculators are OK.
You may use one 8.5 by 11 sheet of handwritten notes.

**Problem 1.** Consider a particle traveling according to the equations

\[ x(t) = \cos^2 t, \quad y(t) = \cos t. \]

Write down and simplify (but do not evaluate) the formula for the length of the curve along which the particle is moving.

**Answer.** Assuming that the particle starts moving at the time \( t = 0 \), we get

\[ L = \int_0^t \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \int_0^t \sqrt{\sin^2(2t) + \sin^2 t} \]

or

\[ L = \int_0^t |\sin t| \sqrt{4 \cos^2 t + 1} \]
**Problem 2.** Consider a particle whose velocity, at time \( t \geq 0 \), is given by

\[
\vec{v}(t) = \langle -2t, -\sin t \rangle
\]

and whose position at \( t = 0 \) is \((4, 0)\).

**a.** Find the formula for the position of the particle at time \( t \).

**b.** Find the point at which the particle crosses the \( y \) axis.

**c.** Suppose the acceleration suddenly drops to 0 at the time when the particle crosses the \( y \)-axis, so that there are no forces acting on the particle. Find the position of the particle one minute later.

**Answers.**

**a.** \( \vec{r}(t) = (-t^2 + 4, \cos(t) - 1) \)

**b.** This happens at \( t = 2 \). The point is \((0, \cos(2) - 1) \simeq (0, -1.4)\)

**c.** When the acceleration drops to zero, the particle continues to move in the direction it was moving, i.e. the velocity is constant. Hence, the position function for \( t \geq 2 \) is \( \vec{r}(2) + (t - 2)\vec{v}(2) = (0, \cos(2) - 1) + (t - 2)(-4, -\sin(2)) = (-4(t - 2), -t\sin(2) + \cos(2) + 2\sin(2) - 1) \simeq (-4(t - 2), -0.9t + 0.4) \). At \( t = 3 \), we get \((-4, -2.3)\).
Problem 3. Find the equations of the normal and of the osculating planes to the curve

\[ \vec{r}(t) = (t^3, \sin(\pi t), t + 1) \]

at the point corresponding to \( t = 2 \).

Answers. \( \vec{r}(2) = (8, 0, 3), \)
\( \vec{r}'(t) = (3t^2, \pi \cos(\pi t), 1) \)
\( \vec{r}'(2) = (12, \pi, 1) \)
Normal plane: \( 12(x - 8) + \pi y + (z - 3) = 0 \)

\[ \vec{r}''(t) = (6t, -\pi^2 \sin(\pi t), 0) \]
\( \vec{r}''(2) = (12, 0, 0) \)
\( \vec{r}' \times \vec{r}'' = (0, 12, -12\pi) \) at the point \( t = 2 \). Hence the normal to the osculating plane can be taken to be \( (0, 1, -\pi) \).
Osculating plane: \( y = \pi(z - 3) \)
Problem 4. Identify the curve
\[ r = 2 \sin \theta + 2 \cos \theta \]
by finding a Cartesian equation for the curve. Give a verbal description of what that curve is.

Answer.
\[ r^2 = 2r \sin \theta + 2r \cos \theta. \]
\[ x^2 + y^2 = 2x + 2y \]
\[ (x - 1)^2 + (y - 1)^2 = 2 \]
Hence this is a circle of radius \( \sqrt{2} \) with the center \((1,1)\).
Problem 5. Consider the function of two variables

\[ f(x, y) = \sqrt{1 + x - y^2}. \]

a. Identify and sketch the domain of \( f(x, y) \).

b. Find the partial derivatives \( f_y(x, y) \) and \( f_x(x, y) \).

c. Find the second partial derivative \( f_{xy}(x, y) \).

d. Find an equation of the tangent plane at the point \((1, 1)\).

Answer.
a. The domain is \( x \geq y^2 - 1 \)

b. \( f_x = \frac{1}{2\sqrt{1+x-y^2}}, \quad f_y = \frac{-y}{\sqrt{1+x-y^2}} \).

c. \( f_{xy} = \frac{uy}{2(1+x-y^2)^{3/2}} \).

d. At the point \((1, 1)\), \( f_x = 1/2, \quad f_y = -1 \) and \( f(1, 1) = 1 \). Hence the equation is \( z - 1 = 1/2(x - 1) - (y - 1) \). Simplifying, we get

\[ x - 2y - 2z = 1 \]