Some answers to practice problems for Midterm I

Math 126, Sections C, D

October, 2006

1. (a) Find the quadratic approximation for the function $f(x) = \ln(x)$ based at e.

   Answer. $T_2(x) = 1 + \frac{x-e}{e} - \frac{(x-e)^2}{2e^2}$

   (b) Use the polynomial from (a) to estimate $\ln(3)$. Compute the error bound for your approximation. Give your answers in both exact and decimal forms.

   Answer. $T_2(3) = 1 + \frac{3-e}{e} - \frac{(3-e)^2}{2e^2} = 1.098$.

   For Taylor's inequality we can take $M = 2/e^3$ which is the maximum of $(\ln(x))'''$ on the interval $[e, 3]$. Hence, $|\ln(3) - T_2(3)| \le \frac{2/e^3}{6}(3-e)^3 = \frac{(3-e)^3}{3e^3} = 0.15$

2. Find Taylor series and the interval of convergence for the following functions

   (a) $f(x) = \frac{1}{(1-x)^3}$ at $b = 0$

   Answer. $\frac{1}{(1-x)^3} = (\frac{1}{2(1-x)})'' = \frac{1}{2}(\sum\limits_0^\infty x^k)'' = \frac{1}{2}(\sum\limits_2^\infty k(k-1)x^{k-2}) =$
   $\sum\limits_0^\infty \frac{(k+1)(k+2)}{2}x^k$. Converges when $|x| < 1$.

   (b) $f(x) = \frac{x^2-3x-4}{(2x-3)(x^2+4)}$ at $b = 0$

   Answer. Using partial fractions, we get

$$\frac{x^2-3x-4}{(2x-3)(x^2+4)} = \frac{x}{x^2+4} - \frac{1}{2x-3}$$

   Now we compute the Taylor series for each summand separately:

   $\frac{x}{x^2+4} = \frac{x}{4}\frac{1}{(1-(-\frac{x^2}{4}))} = \frac{x}{4}\sum\limits_0^\infty(-1)^k\frac{x^{2k}}{4^k} = \sum\limits_0^\infty(-1)^k\frac{x^{2k+1}}{4^{k+1}}$, $|x| < 2$

   $-\frac{1}{2x-3} = \frac{1}{3(1-\frac{2}{3}x)} = \sum\limits_0^\infty \frac{2^k x^k}{3^{k+1}}$, $|x| < 3/2$

   Hence, $\frac{x^2-3x-4}{(2x-3)(x^2+4)} = \sum\limits_0^\infty(-1)^k\frac{x^{2k+1}}{4^{k+1}} + \sum\limits_0^\infty \frac{2^k x^k}{3^{k+1}}$, interval of convergence $(-\frac{3}{2}, \frac{3}{2})$

   (c) $f(x) = e^{3x-2}$ at $b = 2$

   Answer. $e^{3x-2} = \sum\limits_0^\infty \frac{e^4 3^k (x-2)^k}{k!}$, converges everywhere.

   (d) $f(x) = 1 - 6x + 2x^{17} - x^{90}$ at $b = 0$.

   Answer. $1 - 6x + 2x^{17} - x^{90}$

(e) $f(x) = \cos^2(x)$ at $b = 0$

Answer. Use double angle formula: $\cos^2(x) = \frac{1+\cos(2x)}{2}$. We get $\cos^2(x) = \frac{1}{2}(1+(1-\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\ldots)) = \frac{1}{2}(2-\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\ldots) = 1+\sum_{1}^{\infty}(-1)^k\frac{2^{2k-1}x^{2k}}{(2k)!}$, converges everywhere.

(f) $f(x) = \cos(x^2)$ at $b = 0$

Answer. $\cos(x^2) = \sum_{0}^{\infty}(-1)^k\frac{x^{4k}}{(2k)!}$, converges everywhere.

3. Let $f(x) = x^2\ln(1+x^3)$.

(a) Find the Taylor series for $f(x)$ based at $b = 0$.

Answer. $f(x) = \sum_{1}^{\infty}(-1)^{k-1}\frac{x^{3k+2}}{k}$

(b) Explicitly compute the coefficient by x^{17} in the Taylor expansion from (a).

Answer. To get the coefficient by x^{17} in the series above, we use $k = 5$ since $3*5+2 = 17$. Computing the coefficient for $k = 5$, we get $(-1)^4/5$.

(c) Find $f^{17}(0)$.

Answer. By the Taylor formula, the coefficient by x^{17} in the series above equals $f^{17}(0)/17!$. Hence, we have the equation $f^{17}(0)/17! = 1/5$. We obtain $f^{17}(0) = 17!/5$.

4. Find the fourth Taylor polynomial based at $b = 0$ of the function $f(x) = \frac{e^{x^2}}{x^2-1}$ without differentiating.

Answer. $T_4(x) = -(1+2x^2+\frac{5}{4}x^4)$

5. Approximate the integral $\int_0^2 \sin(x^2)dx$ using the first three non-zero terms of the Taylor series.

Answer. First, we write the first 3 non-zero terms of the Taylor series for $\sin(x^2)$:

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$$

and now integrate
$\int_0^2 (x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!})dx = (\frac{x^3}{3} - \frac{x^7}{7*3!} + \frac{x^{11}}{11*5!})\Big|_0^2 = \frac{8}{3} - \frac{128}{42} + \frac{2048}{1320} = 1.17$

6. Let $\bar{u} = (3, 2, -1)$, $\bar{v} = (2, -2, 2)$. Find a unit vector $\bar{w}$ perpendicular to both $\bar{u}$ and $\bar{v}$.

   Answer. $(\frac{1}{\sqrt{42}}, -\frac{4}{\sqrt{42}}, -\frac{5}{\sqrt{42}})$ or $(-\frac{1}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}})$

7. (a) Check if the points $A = (5, 1, 3)$, $B = (7, 9, -1)$, $C = (1, -15, 11)$ are colinear in TWO different ways.

   Answer. Yes.

   (b) Now let $A = (1, 2, -3)$, $B = (3, 4, -2)$, $C = (3, -2, 1)$. Check whether the triangle ABC has an obtuse angle. Find the area of the triangle.

   Answer. This is a right triangle, the area is 9.

8. (a) Show that the equation

$$x^2 + y^2 + z^2 = 4x + z$$

   represents a sphere, and find its center and radius.

   Answer. Completing squares, we get $x^2 + y^2 + z^2 - 4x - z = (x-2)^2 + y^2 + (z - 1/2)^2 - 4 - 1/4$. Hence, the equation of the sphere is

$$(x-2)^2 + y^2 + (z - 1/2)^2 = 17/4$$

   Center: $(2, 0, 1/2)$, radius $R = \sqrt{17}/2$.

   (b) Check that the point $A = (4, 0, 1)$ is on the sphere from (a). Then show that the point $B = (3, 2, 5)$ belongs to the plane tangent to the sphere at the point A.

   Hint: Tangent plane to the sphere at the point A is perpendicular to the radius connecting the center to the point A.