1. (9pts) A curve is given by the equation
\[ r + \frac{1}{r} = 4 \cos \theta \]
in polar coordinates.

(a) (7pts) Find an equation of the curve in Cartesian coordinates. Sketch the curve.

\textbf{Solution.}
Multiplying both sides by \( r \), we get
\[ r^2 + 1 = 4r \cos \theta \]
\[ x^2 + y^2 + 1 = 4x \]
\[ (x - 2)^2 + y^2 = 3 \]
Hence the curve is a circle with the center \((2, 0)\) and radius \( \sqrt{3} \).

(b) (2pts) Using your sketch, find all the points on the curve where the tangent line is horizontal.

\textbf{Solution.} Since the curve is a circle, the tangent line is horizontal at the points right above and right below the center: \((2, \sqrt{3})\) and \((2, -\sqrt{3})\).
2. (15pts) Consider the ellipse given by the equation

\[
\frac{(x-1)^2}{9} + \frac{(y-1)^2}{16} = 1
\]

(a) (5pts) Find parametric equations of the ellipse.

**Solution.** \((\frac{x-1}{3})^2 + (\frac{y-1}{4})^2 = 1\). Hence, set \(\frac{x-1}{3} = \cos t, \frac{y-1}{4} = \sin t\). Solving for \(x, y\), we get

\[x = 3 \cos t + 1, \quad y = 4 \sin t + 1\]

(b) (5pts) Set up, but do not evaluate, the integral computing the length of the arc of the ellipse going clockwise from the point \((1, 5)\) to the point \((4, 1)\).

**Answer.** \(\int_{\pi/2}^{\pi} \sqrt{9 \sin^2(t) + 16 \cos^2(t)} dt\)

(c) (5pts) Using the sketch of the ellipse, find the points where the curvature of the ellipse is minimal. Compute the value of the curvature at these points.

**Answer.** The curvature is minimal at the most left and most right points of the ellipse: \((-2, 1)\) and \((4, 1)\), corresponding to \(t = 0\) and \(t = \pi\). The curvature at these two points is the same, so we shall compute \(\kappa\) at \(t = 0\).

We shall use the formula

\[\kappa = \frac{||r' \times r''||}{||r'||^3}\]

for the curvature. We compute

\[r'(t) = (-3 \sin t, 4 \cos t), \quad r''(t) = (-3 \cos t, -4 \sin t)\]

Hence, \(r'(0) = (0, 4), r''(0) = (-3, 0)\). We can in particular observe that \(r'(0)\) and \(r''(0)\) are perpendicular (take the dot product). Hence,

\[||r'(0) \times r''(0)|| = ||r'||||r''|| \sin(\pi/2) = 4 \cdot 3 \cdot 1 = 12\]

Since \(||r'(0)|| = 4\), the formula (*) implies

\[\kappa = \frac{12}{4^3} = \frac{3}{16}\]
3. (12pts) A particle is moving with the acceleration \( \vec{a}(t) = (-\cos t, -\sin t, 0) \), \( t \geq 0 \). The initial position of the particle is \((1, 0, 0)\) and the initial velocity is \((0, 1, 1)\)

(a) (6pts) Find the position vector \( \vec{r}(t) \).

**Solution.** \( \vec{v}(t) = (-\sin t, \cos t, 1) \), \( \vec{r}(t) = (\cos t, \sin t, t) \).

(b) (6pts) Find the length of the projection of the acceleration vector onto the unit tangent vector.

**Solution.** \( T = \frac{\vec{r}'}{||\vec{r}'||} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \). Hence,

\[
a_T = \text{comp}_T \vec{a} = \frac{\vec{a} \cdot T}{||T||} = \vec{a} \cdot T = (-\cos t, -\sin t, 0) \cdot \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) = 0.
\]

Alternatively, we can use the formula \( a_T = v' \). Here, \( v = ||\vec{r}'(t)|| = \sqrt{2} \). Hence, \( a_T = v' = 0 \).

4. (12pts) Let \( f(x, y) = \sqrt{11 - x^2 - y^2} \).

(a) (3pts) Sketch the contour map of the function \( f \) consisting of level curves for the level \( k = 0, 1, 2, 3 \). Label the curves.

(b) (3pts) Compute \( f_x(x, y) \).

**Answer.** \( f_x(x, y) = -\frac{x}{\sqrt{11 - x^2 - y^2}} \)

(c) (3pts) Compute \( f_y(x, y) \).

**Answer.** \( f_y(x, y) = -\frac{y}{\sqrt{11 - x^2 - y^2}} \)

(d) (3pts) Find an equation of the tangent plane to the graph of \( f \) at the point \((1, 1, 3)\).

**Answer.** \( f_x(1,1) = -1/3, f_y = -1/3 \). Hence

\[
z - 3 = -\frac{1}{3}(x - 1) - \frac{1}{3}(y - 1)
\]

or

\[
x + y + 3z = 11
\]
5. (12pts) Consider a curve given by the parametric equations $(t, t^2 - 1, t^3 + 1)$.

(a) (8pts) Find an equation of the normal plane at the point $(0, -1, 1)$.

Solution. $r'(t) = (1, 2t, 3t^2)$. The point $(0, -1, 1)$ corresponds to $t = 0$. The normal vector to the normal plane is the tangent vector. To get the tangent vector at the point $(0, -1, 1)$ we have to plug in $t = 0$ into $r'(t) = (1, 2t, 3t^2)$. (Caution: we do not plug in the point $(0, -1, 1)$ itself anywhere, we find the corresponding value of $t$, and then plug it in). We get $r'(0) = (1, 0, 0)$. The equation of the normal plane is

$x = 0$

(b) (4pts) Find the angle between the normal and osculating planes at the point $(1, 0, 2)$.

Solution. The normal vector to the normal plane is $T$, and to the osculating plane is $B$. Since $T$ is perpendicular to $B$ (always!), the angle between the two planes is $\pi/2$.

6.(Bonus) (2pts) Let $\vec{r}(t) = (\cos t \sin t, \sin^2 t, \cos t)$. Show that $\vec{r}'$ is perpendicular to $\vec{r}$ at any point on the curve WITHOUT computing $\vec{r}'$.

Solution. Since $r'(t)$ is perpendicular to $r(t)$ whenever $r(t)$ has constant length, it suffices to show that the given vector function $r$ the length is independent of $t$, i.e. a constant. We compute

$$||r(t)|| = \sqrt{\cos^2 t \sin^2 t + \sin^4 t + \cos^2 t} = \sqrt{\sin^2 t (\cos^2 t + \sin^2 t) + \cos^2 t} = \sqrt{\sin^2 t + \cos^2 t} = 1$$

We used the identity $\cos^2 t + \sin^2 t = 1$ twice in this calculation.