Sample MIDTERM II, version 2.
Last year's midterm for Spring MATH 126 C, D

Scientific, but not graphing calculators are OK.

You may use one 8.5 by 11 sheet of handwritten notes.

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1. Find the slope of the tangent line to the polar curve

$$r = \frac{1}{\theta}, \theta > 0$$

at the point where it intersects the cartesian curve

$$x^2 + y^2 = \frac{1}{9}.$$

Answer. To find the point of intersection, we solve $x^2 + y^2 = r^2 = 1/9$
$r = 1/3$
$1/\theta = 1/3$
$\theta = 3$
Using the polar-Cartesian conversion formulas, we get $x = r\cos\theta = \frac{1}{3}\cos(3), y = r\cos\theta = \frac{1}{3}\sin(3)$
Hence, the point is $(\frac{1}{3}\cos(3), \frac{1}{3}\sin(3))$.
For the tangent line, we need to find the slope dy/dx at the point $\theta = 3$. We have
$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{d}{d\theta}(\sin(\theta)/\theta)}{\frac{d}{d\theta}(\cos(\theta)/\theta)} = \frac{\theta\cos(\theta)-\sin(\theta)}{-\theta\sin(\theta)-\cos(\theta)} = \frac{\sin(3)-3\cos(3)}{3\sin(3)+\cos(3)} = \frac{\tan(3)-3}{3\tan(3)+1}$.
Tangent line: $y - \frac{\sin(3)}{3} = \frac{\tan(3)-3}{3\tan(3)+1}(x - \frac{\cos(3)}{3})$

2. At what point(s) is the tangent line to the curve

$$x = t^3 - 3t, y = t^2 + 2t$$

parallel to the line with parametric equations

$$x = 3s + 5, y = s - 6 \text{ ?}$$

Answer. The direction of the tangent line is the tangent vector $(3t^2 - 3, 2t + 2)$. The direction of the line is $(3, 1)$. Hence we have to find t such that $(3t^2 - 3, 2t + 2)$ is parallel to $(3, 1)$. Hence, we have to solve

$$(3t^2 - 3, 2t + 2) = a(3, 1)$$

for some number a. We get two equations:
$3t^2 - 3 = 3a$
$2t + 2 = a$
Solving for t and a, we obtain $t = 3$, $a = 8$. Plugging in $t = 3$ into the parametric equations, we obtain that the point is $(18, 15)$.

Alternatively, the slope of the tangent line is $\frac{dy}{dx} = \frac{2t+2}{3t^2-3}$, and the slope of the other line is $1/3$. Hence, we need to solve

$$\frac{2t + 2}{3t^2 - 3} = \frac{1}{3}$$

There is a unique solution $t = 3$. The corresponding point is $(18, 15)$.

Remark. The advantage of the first method is that it works equally well in three dimensional situation.

3. For any $m > 0$, the helix determined by the position function

$$\vec{r}(t) = \langle \cos t, \sin t, mt \rangle$$

has constant curvature that depends on m. Find the value of m such that the curvature at any point on the curve is $\frac{1}{3}$.

Answer. The curvature is $\kappa = \frac{1}{m^2+1}$. Solving $\frac{1}{m^2+1} = 1/3$, we get $m = \sqrt{2}$.

4. A particle is moving so that its position is given by the vector function

$$\vec{r}(t) = \langle t^2, t, 5t \rangle$$

Find the tangent and normal components of the particle's acceleration vector.

Answer. $\vec{r}' = (2t, 1, 5)$, $v = ||\vec{r}'|| = \sqrt{26+4t^2}$
$T = \frac{1}{\sqrt{26+4t^2}}(2t, 1, 5)$
$\vec{a} = \vec{r}'' = (2, 0, 0)$.

We use the notation a_T for the tangential component, and a_N fpr the normal component; that is a_T is the length of the projection of $\vec{a}$ on the Tangent vector (T), and a_N is the length of the projection of $\vec{a}$ onto the Normal vector (N). In particular, a_T, a_N are both scalars.

The tangential component of $\vec{a}$ is the length of the projection of $\vec{a}$ onto T. Hence,

$$a_T = \frac{\vec{a} \cdot T}{||T||} = \vec{a} \cdot T = (2, 0, 0) \cdot \frac{1}{\sqrt{26+4t^2}}(2t, 1, 5) = \frac{\mathbf{4t}}{\sqrt{\mathbf{26+4t^2}}}$$

We can compute a_N in several different ways.

I. Use the formula

$$a_N = \frac{||\vec{a} \times \vec{v}||}{||\vec{v}||} = \frac{||(2,0,0) \times (2t,1,5)||}{\sqrt{26+4t^2}} = \frac{||(0,-10,2)||}{\sqrt{26+4t^2}} = \sqrt{\frac{104}{26+4t^2}} = \frac{\mathbf{2\sqrt{13}}}{\sqrt{\mathbf{13+2t^2}}}$$

II. Another way is to notice that the decomposition

$$\vec{a} = a_T T + a_N N$$

implies that

$$a_N N = \vec{a} - a_T T$$

Since N is a unit vector, we get

$$a_N = ||\vec{a} - a_T T||$$

And we already know a_T so we can just plug it in! We now compute

$a_N = ||\vec{a} - a_T T|| = ||(2,0,0) - \frac{4t}{\sqrt{26+4t^2}} \frac{(2t,1,5)}{\sqrt{26+4t^2}}|| = ||(2,0,0) - \frac{(8t^2,4t,20t)}{26+4t^2}|| =$

$||(2 - \frac{8t^2}{26+4t^2}, -\frac{4t}{26+4t^2}, -\frac{20t}{26+4t^2})|| = ||(\frac{52}{26+4t^2}, -\frac{4t}{26+4t^2}, -\frac{20t}{26+4t^2})|| =$
$\frac{4}{26+4t^2}||(13, -t, -5t)|| = \frac{2}{13+2t^2}\sqrt{13^2 + t^2 + 25t^2} = \frac{2\sqrt{13(13+2t^2)}}{13+2t^2} = \frac{2\sqrt{13}}{\sqrt{13+2t^2}} = \sqrt{\frac{52}{13+2t^2}}$.

III. Yet another way is to use the Pythagoras theorem:

$$||\vec{a}||^2 = a_T^2 + a_N^2$$

Hence, $a_N = \sqrt{||\vec{a}||^2 - a_T^2} = \sqrt{4 - \frac{16t^2}{26+4t^2}} = \sqrt{\frac{104}{26+4t^2}} = \sqrt{\frac{52}{13+2t^2}}$

5. Reparametrize the curve

$$\vec{r}(t) = \langle 5t - 1, 2t, 3t + 2 \rangle$$

with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

Answer. $||r'|| = \sqrt{38}$. Take $s = \sqrt{38}t$. Then $\vec{r}(s) = \langle 5s/\sqrt{38} - 1, 2s/\sqrt{38}, 3s/\sqrt{38} + 2 \rangle$ is a natural parameterization.

6. Let $f(x,y) = x^2y + x\sin y - \ln(x - y^2)$.

(a) Find $f_y(x,y)$.

Answer. $f_y(x,y) = x^2 + x\cos y + \frac{2y}{x-y^2}$

(b) Find $f_{xy}(x,y)$.

Answer. $f_{xy}(x,y) = f_{yx}(x,y) = 2x + \cos y - \frac{2y}{(x-y^2)^2}$