Solutions to Math 126A Quiz 2

1. Here is the Taylor series for the function $f(x) = \cos x$ based at $a = 0$:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

   (a) Write the 12^{th} Taylor polynomial for $f(x) = \cos x$ based at $a = 0$.
   Solution:

$$T_{12} = \sum_{n=0}^{6} \frac{(-1)^n}{(2n)!} x^{2n},$$

   or

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}.$$

   Notice the upper limit on the summation is **6** and not 12 in order to get a polynomial of degree 12!

   (b) Find n such that the error $|\cos x - T_n(x)|$ is at most 0.01 on the interval $[-1, 1]$.
   Solution: We use the Taylor inequality to obtain

$$|\cos x - T_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1},$$

   where M is the largest value that $|f^{(n+1)}(x)|$ can take on $[-1, 1]$. But on $[-1, 1]$, $|x| \leq 1$ and so $|x|^{n+1} \leq 1$. Also, the derivatives of $\cos x$ cycle through $\pm \sin x$ and $\pm \cos x$ and each of these functions is bounded by 1 on the entire real line. Consequently,

$$|\cos x - T_n(x)| \leq \frac{1}{(n+1)!}.$$

   In order to force the error to be less than $0.01 = 1/100$, we therefore take

$$\frac{1}{(n+1)!} < \frac{1}{100}.$$

   Since $5! = 120$, we find that $n = 4$ works, as does any $n \geq 4$.