NAME ______________________ SECTION ________

# Practice problems for the Final

Math 126, Section A

Material covered after Midterm II

1. Find and classify critical points of the function

   (a) $f(x, y) = xy^2 - 2x^2 - y^2$

   **Solution.**

   $f_x = y^2 - 4x = 0$
   $f_y = 2xy - 2y = 0$
   Solving, we get 3 critical points: $(0, 0)$, $(1, 2)$, $(1, -2)$.

   To classify the critical points we have to use the Second Derivative test. We compute

   $$f_{xx} = -4 \quad f_{yy} = 2x - 2 \quad f_{xy} = 2y$$

   Hence, $D = f_{xx}f_{yy} - f_{xy}^2 = -4(2x - 2) - 4y^2 = 8 - 8x - 8y^2$.

   At the critical point $(0, 0)$, $D = 8$, $f_{xx} = -4$. Hence $(0, 0)$ is a local *maximum.*

   At the critical points $(1, 2), (1, -2)$, $D = -32$. Hence, $(1, 2), (1, -2)$ are *saddle* points.

   (b) $f(x, y) = 3xy - x^2y - xy^2$

   **Solution.**

   $f_x = 3y - 2xy - y^2 = 0$
   $f_y = 3x - x^2 - 2xy = 0$
   Subtracting, we get
   $3(y - x) = y^2 - x^2 \quad \Rightarrow$
   $3(y - x) = (y - x)(y + x) \quad \Rightarrow$
   $y = x$ or $y + x = 3$
   If $y = x$, then plugging into the first equation we get $y = 0$ or $y = 1$. Hence we obtain two critical points in this case: $(0, 0)$ and $(1, 1)$
   If $y + x = 3$, the plugging $x = 3 - y$ into the first equation, we get $y = 0$ and $y = 3$. Hence, we obtain two new critical points: $(3, 0)$ and $(0, 3)$.

   To classify the critical points we have to use the Second Derivative test. We compute

   $$f_{xx} = -2y, \quad f_{yy} = -2x, \quad f_{xy} = 3 - 2x - 2y$$

Hence, $D = f_{xx}f_{yy} - f_{xy}^2 = 4xy - (3 - 2x - 2y)^2$.

At the critical point $(0, 0)$, $D = -9$. Hence $(0, 0)$ is a *saddle* point.

At the critical point $(1, 1)$, $D = 3$, $f_{xx} = -2$. Hence, $(1, 1)$ is a local *maximum.*

At the critical point $(3, 0)$, $D = -9$. Hence, $(3, 0)$ is a *saddle.* Since the equation is symmetric in $x$ and $y$, we conclude that $(0, 3)$ is also a *saddle.*

2. Find the points on the surface $xy^2z^3 = 1$ which are closest to the origin.

**Solution.** We have to minimize the function $f = x^2 + y^2 + z^2$ where $(x, y, z)$ are points on the surface given by the equation $xy^2z^3 = 1$. Hence, we have to solve

$$f_x = 2x + 2zz_x = 0$$

$$f_y = 2y + 2zz_y = 0$$

We can find $z_x, z_y$ by implicitly differentiating the equation of the surface $xy^2z^3 = 1$. Applying $\frac{\partial}{\partial x}$ and using the product rule, we get

$$y^2z^3 + 3xy^2z^2z_x = 0 \qquad (*)$$

Now applying $\frac{\partial}{\partial y}$ and using the product rule again, we get

$$2xyz^3 + 3xy^2z^2z_y = 0 \qquad (**)$$

None of the $x, y, z$ can be zero since $(x, y, z)$ is a point on the surface $xy^2z^3 = 1$. Solving the equation (*) for $z_x$ and the equation (**) for $z_y$, we get

$$z_x = -\frac{z}{3x}, \quad z_y = -\frac{2z}{3y}$$

Now plug $z_x$, $z_y$ into equations for $f_x$, $f_y$. We get

$$x + z(-\frac{z}{3x}) = 0, \quad y + z(-\frac{2z}{3y}) = 0$$

Hence,

$$3x^2 = z^2, \quad 3y^2 = 2z^2$$

Finally, plugging $x = \pm\frac{z}{\sqrt{3}}$, $y^2 = \frac{2z^2}{3}$ into the equaion of the surface, we get

$$\pm\frac{z}{\sqrt{3}}\frac{2z^2}{3}z^3 = \pm\frac{2z^6}{3^{3/2}} = 1$$

The "$-$" sign is not possible, and solving for $z$ we get $z = \pm 3^{\frac{1}{4}}2^{-\frac{1}{6}} = \pm\frac{\sqrt[4]{3}}{\sqrt[6]{2}}$. Hence, the closest points are

$$(3^{-\frac{1}{4}}2^{-\frac{1}{6}}, \pm 3^{-\frac{1}{4}}2^{\frac{1}{3}}, 3^{\frac{1}{4}}2^{-\frac{1}{6}}), (-3^{-\frac{1}{4}}2^{-\frac{1}{6}}, \pm 3^{-\frac{1}{4}}2^{\frac{1}{3}}, -3^{\frac{1}{4}}2^{-\frac{1}{6}})$$

3. (a) Reverse the order of integration and then evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dxdy$$

**Solution.** The integral in reversed order is

$$\int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dydx$$

(You need to draw a picture of the region $D$ given by $\{(x,y) : 0 \le y \le 1, \sqrt{y} \le x \le 1\}$ to find , integration limits in the reversed order)

Now, integrate

$\int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dydx = \int_0^1 \left(y\sqrt{x^3+1}\right)\Big|_0^{x^2} dx = \int_0^1 x^2\sqrt{x^3+1}\, dx = \frac{2(x^3+1)^{\frac{3}{2}}}{9}\Big|_0^1 = \frac{4\sqrt{2}}{9} - \frac{2}{9}$

(b) Evaluate the following integral

$$\int_0^1 \int_{x^2}^1 x\sin(\pi y^2)\, dydx$$

**Solution.** We reverse the order of integration first.

$$\int_0^1 \int_0^{\sqrt{y}} x\sin(\pi y^2)\, dxdy$$

and then evaluate

$\int_0^1 \int_0^{\sqrt{y}} x\sin(\pi y^2)\, dxdy = \int_0^1 \left(\frac{x^2}{2}\sin(\pi y^2)\Big|_0^{\sqrt{y}}\right) dy = \frac{1}{2}\int_0^1 y\sin(\pi y^2)\, dy = -\frac{\cos(\pi y^2)}{4\pi}\Big|_0^1 = \frac{1}{2\pi}$

4. Find the volume of the solid bounded by the cylinder $x^2+y^2=1$ and the planes $y=z$, $x=0$, $z=0$ in the first octant.
*Do this problem in two ways: using rectangular coordinates, and then using polar coordinates.*

**Solution.** The region here is 1/4 of the circle $x^2+y^2=1$, the quarter in the first quadrant. The function is $z=y$. Hence, we need to evaluate

$$\int\int_D ydxdy$$

I. Cartesian coordinates.

$$\int\int_D ydxdy = \int_0^1 \int_0^{\sqrt{1-x^2}} y\, dydx = \int_0^1 \frac{y^2}{2}\Big|_0^{\sqrt{1-x^2}} dx = \frac{1}{2}\int_0^1 (1-x^2)dx = \frac{1}{3}$$

II. Polar coordinates.

$$\int\int_D y dx dy = \int_0^{\frac{\pi}{2}}\int_0^1 (r\sin\theta) r\, dr d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 (\sin\theta) r^2\, dr d\theta = \left(\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta\right)\left(\int_0^1 r^2 dr\right) = \frac{1}{3}$$

5. Compute the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ from below and $z = \frac{x^2}{2} + \frac{y^2}{2} + 1$ from above.

**Solution.** First, find the intersection of two paraboloids:
$x^2 + y^2 = \frac{x^2}{2} + \frac{y^2}{2} + 1$
$x^2 + y^2 = 2$
This is the equation of the projection of the intersection onto the $xy$-plane. The solid in questions lies above this circle, so we take

$$D = \{(x, y) : x^2 + y^2 = 2\}$$

The integral computing the volume of the solid is

$$V = \int\int_D (\frac{x^2}{2} + \frac{y^2}{2} + 1 - x^2 - y^2) dx dy = \int\int_D (1 - \frac{x^2}{2} - \frac{y^2}{2}) dx dy$$

Changing to polar coordinates, we obtain

$$\int_0^{2\pi}\int_0^{\sqrt{2}} (1 - \frac{r^2}{2}) r dr d\theta = \pi$$

6. Evaluate the double integral

$$\int\int_D (x^2 + x + y^2)\, dA$$

where $D$ is the region

$$D = \{(x, y) : x^2 + y^2 \leq 4 \,\text{and}\, \text{y} \geq \text{x}\}$$

**Solution.** In polar coordinates

$$D = \{(r, \theta) : r \leq 2 \,\text{and}\, \frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4}\}$$

Hence, the integral in polar coordinates is

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\int_0^2 (r^2 + r\cos\theta) r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\int_0^2 (r^3 + r^2\cos\theta) dr d\theta = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\frac{r^4}{4} + \frac{r^3}{3}\cos\theta)\Big|_0^2 d\theta =$$

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (4 + \frac{8}{3}\cos\theta) d\theta = 4\theta + \frac{8}{3}\sin\theta\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = 4\pi - \frac{8\sqrt{2}}{3}$$