Some answers to practice problems for Midterm I

Math 126, Section A
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1. (a) Find the quadratic approximation for the function $f(x) = \ln(x)$ based at e.

   Answer. $T_2(x) = 1 + \frac{x-e}{e} - \frac{(x-e)^2}{2e^2}$

   (b) Use the polynomial from (a) to estimate $\ln(3)$. Compute the error bound for your approximation. Give your answers in both exact and decimal forms.

   Answer. $T_2(3) = 1 + \frac{3-e}{e} - \frac{(3-e)^2}{2e^2} = 1.098$.

   For Taylor's inequality we can take $M = 2/e^3$ which is the maximum of $(\ln(x))'''$ on the interval $[e, 3]$. Hence, $|\ln(3) - T_2(3)| \leq \frac{2/e^3}{6}(3-e)^3 = \frac{(3-e)^3}{3e^3} = 0.15$

2. Let $f(x) = e^x$, and let $T_n(x)$ be the n^{th} Taylor polynomial for $f(x)$ based at $a = 0$. Find n such that the error $|T_n(x) - e^x| \leq 0.001$ on the interval $I = [-1, 1]$.

   Answer. $T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots + \frac{x^n}{n!}$. We also have the Taylor's inequality

$$|T_n - e^x| \leq \frac{M}{(n+1)!}|x|^{n+1}$$

   where $|f^{n+1}(x)| \leq M$ on the interval I. Since $f^{n+1}(x) = e^x$, and the interval is $[-1, 1]$, we can take $M = e$. Hence, the error for $|T_n(x) - e^x|$ is bounded by $\frac{e}{(n+1)!}|x|^n$. On the interval $[-1, 1]$ we have $|x^n| \leq 1$, and hence the error is further bounded by $\frac{e}{(n+1)!}$. Thus, we need to find n such that

$$\frac{e}{(n+1)!} \leq 0.001.$$

   Using "trial-and-error" method (i.e. plugging in $n = 1, 2, \ldots$ into the formula for the error) we find that $n \geq 6$ works.

3. Find Taylor series and the interval of convergence for the following functions

   (a) $f(x) = \frac{1}{(1-x)^3}$ at $b = 0$

   Answer. $\frac{1}{(1-x)^3} = (\frac{1}{2(1-x)})'' = \frac{1}{2}(\sum\limits_0^\infty x^k)'' = \frac{1}{2}(\sum\limits_2^\infty k(k-1)x^{k-2}) = \sum\limits_0^\infty \frac{(k+1)(k+2)}{2}x^k$. Converges when $|x| < 1$.

   (b) $f(x) = \frac{x^2-3x-4}{(2x-3)(x^2+4)}$ at $b = 0$

   Answer. Using partial fractions, we get

$$\frac{x^2 - 3x - 4}{(2x-3)(x^2+4)} = \frac{x}{x^2+4} - \frac{1}{2x-3}$$

Now we compute the Taylor series for each summand separately:

$\frac{x}{x^2+4} = \frac{x}{4}\frac{1}{(1-(-\frac{x^2}{4}))} = \frac{x}{4}\sum_0^\infty(-1)^k\frac{x^{2k}}{4^k} = \sum_0^\infty(-1)^k\frac{x^{2k+1}}{4^{k+1}}$, $|x| < 2$

$-\frac{1}{2x-3} = \frac{1}{3(1-\frac{2}{3}x)} = \sum_0^\infty \frac{2^k x^k}{3^{k+1}}$, $|x| < 3/2$

Hence, $\frac{x^2-3x-4}{(2x-3)(x^2+4)} = \sum_0^\infty(-1)^k\frac{x^{2k+1}}{4^{k+1}} + \sum_0^\infty \frac{2^k x^k}{3^{k+1}}$, interval of convergence $(-\frac{3}{2}, \frac{3}{2})$

(c) $f(x) = e^{3x-2}$ at $b = 2$

Answer. $e^{3x-2} = \sum_0^\infty \frac{e^4 3^k (x-2)^k}{k!}$, converges everywhere.

(d) $f(x) = 1 - 6x + 2x^{17} - x^{90}$ at $b = 0$.

Answer. $1 - 6x + 2x^{17} - x^{90}$

(e) $f(x) = \cos^2(x)$ at $b = 0$

Answer. Use double angle formula: $\cos^2(x) = \frac{1+\cos(2x)}{2}$. We get $\cos^2(x) = \frac{1}{2}(1+(1-\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\ldots)) = \frac{1}{2}(2-\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\ldots) = 1+\sum_1^\infty(-1)^k\frac{2^{2k-1}x^{2k}}{(2k)!}$, converges everywhere.

(f) $f(x) = \cos(x^2)$ at $b = 0$

Answer. $\cos(x^2) = \sum_0^\infty(-1)^k\frac{x^{4k}}{(2k)!}$, converges everywhere.

(g) $f(x) = xe^x$ at $a = 2$

Answer. First, make substitution $u = x - 2$. Hence, $x = u + 2$, and we have

$$f(u) = (u+2)e^{u+2} = ue^{u+2} + 2e^{u+2} = ue^u e^2 + 2e^u e^2 = e^2 ue^u + 2e^2 e^u$$

Now, e^2 and $2e^2$ are constants. We can apply the series for e^u to both summands:

$$f(u) = e^2 u\sum_0^\infty \frac{u^n}{n!} + 2e^2\sum_0^\infty \frac{u^n}{n!} = \sum_0^\infty e^2\frac{u^{n+1}}{n!} + \sum_0^\infty 2e^2\frac{u^n}{n!}$$

Plugging in $u = x - 2$, we get the series in terms of $x - 2$:

$$f(x) = \sum_0^\infty e^2\frac{(x-2)^{n+1}}{n!} + \sum_0^\infty 2e^2\frac{(x-2)^n}{n!}.$$

The series converges everywhere.

Although this is not required here is the answer written with only one summation sign:

$$f(x) = 2e^2 + \sum_1^\infty e^2(\frac{1}{(n-1)!} + \frac{2}{n!})(x-2)^n$$

(h) $f(x) = \int\limits_0^x \frac{e^t-1}{t}dt$

Solution. $e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots$, hence
$e^t - 1 = t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots$, hence

$$\frac{e^t-1}{t} = 1 + \frac{t}{2} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots = \sum_0^\infty \frac{t^n}{(n+1)!},$$

converges everywhere. Integrating, we get

$$f(x) = \int\limits_0^x \frac{e^t-1}{t}dt = \int\limits_0^x 1+\frac{t}{2}+\frac{t^2}{3!}+\frac{t^3}{4!}+\ldots dt = x+\frac{x^2}{2*2}+\frac{x^3}{3*3!}+\frac{x^4}{4*4!}+\ldots+\frac{x^n}{n*n!}+\ldots$$

Alternatively, switching to the summation notation right away, we have

$$f(x) = \int\limits_0^x \frac{e^t-1}{t}dt = \int\limits_0^x \sum_0^\infty \frac{t^n}{(n+1)!}dt = \sum_0^\infty \int_0^x \frac{t^n}{(n+1)!}dt = \sum_0^\infty \frac{x^{n+1}}{(n+1)(n+1)!} = \sum_1^\infty \frac{x^n}{n(n!)}$$

4. Let $f(x) = x^2 \ln(1 + x^3)$.

   (a) Find the Taylor series for $f(x)$ based at $b = 0$.

   Answer. $f(x) = \sum\limits_1^\infty (-1)^{k-1} \frac{x^{3k+2}}{k}$

   (b) Explicitly compute the coefficient by x^{17} in the Taylor expansion from (a).

   Answer. To get the coefficient by x^{17} in the series above, we use $k = 5$ since $3 * 5 + 2 = 17$. Computing the coefficient for $k = 5$, we get $(-1)^4/5$.

   (c) Find $f^{17}(0)$.

   Answer. By the Taylor formula, the coefficient by x^{17} in the series above equals $f^{17}(0)/17!$. Hence, we have the equation $f^{17}(0)/17! = 1/5$. We obtain $f^{17}(0) = 17!/5$.

5. Find the fourth Taylor polynomial based at $b = 0$ of the function $f(x) = \frac{e^{x^2}}{x^2-1}$ without differentiating.

   Answer. $T_4(x) = -(1 + 2x^2 + \frac{5}{2}x^4)$

6. Approximate the integral $\int\limits_0^2 \sin(x^2)dx$ using the first three non-zero terms of the Taylor series.

   Answer. First, we write the first 3 non-zero terms of the Taylor series for $\sin(x^2)$:

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$$

   and now integrate
   $\int\limits_0^2 (x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!})dx = (\frac{x^3}{3} - \frac{x^7}{7*3!} + \frac{x^{11}}{11*5!})\Big|_0^2 = \frac{8}{3} - \frac{128}{42} + \frac{2048}{1320} \simeq 1.17$

8. Let $\bar{u} = (3, 2, -1)$, $\bar{v} = (2, -2, 2)$. Find a unit vector $\bar{w}$ perpendicular to both $\bar{u}$ and $\bar{v}$.

   Answer. $(\frac{1}{\sqrt{42}}, -\frac{4}{\sqrt{42}}, -\frac{5}{\sqrt{42}})$ or $(-\frac{1}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}})$

9. (a) Check if the points $A = (5, 1, 3)$, $B = (7, 9, -1)$, $C = (1, -15, 11)$ are colinear in TWO different ways.

   Answer. Yes.

   (b) Now let $A = (1, 2, -3)$, $B = (3, 4, -2)$, $C = (3, -2, 1)$. Check whether the triangle ABC has an obtuse angle. Find the area of the triangle.

   Answer. This is a right triangle, the area is 9.

10. (a) Show that the equation

$$x^2 + y^2 + z^2 = 4x + z$$

   represents a sphere, and find its center and radius.

   Answer. Completing squares, we get $x^2 + y^2 + z^2 - 4x - z = (x-2)^2 + y^2 + (z - 1/2)^2 - 4 - 1/4$. Hence, the equation of the sphere is

$$(x-2)^2 + y^2 + (z - 1/2)^2 = 17/4$$

   Center: $(2, 0, 1/2)$, radius $R = \sqrt{17}/2$.

   (b) Check that the point $A = (4, 0, 1)$ is on the sphere from (a). Then show that the point $B = (3, 2, 5)$ belongs to the plane tangent to the sphere at the point A.

   Hint: Tangent plane to the sphere at the point A is perpendicular to the radius connecting the center to the point A.