Name ______________________________ Quiz Section __________

MIDTERM II

Math 126, Section A

February 22, 2007

Problem	Total Points	Score
1	12	
2	12	
3	12	
4	14	
Total	50	
5(Bonus)	3	

- You may use a scientific calculator and one one-sided sheet of handwritten notes. No other notes, books or calculators are allowed. Please turn off your cell phone.

- Show all your work to get full credit.

- Read instructions for each problem CAREFULLY.

- Leave all your answers in EXACT form.

- Check your work!

1. (10pts) Consider the curve given by the equation in polar coordinates

$$r = 4\cos\theta + \sin\theta.$$

(a)(5pts) Find the Cartesian equation of the curve. Sketch the curve.

Solution. Multuply the equation by r:
$r^2 = 4r\cos\theta + r\sin\theta$
Hence, $x^2 + y^2 = 4x + y$
$x^2 - 4x + y^2 - y = 0$
$(x-2)^2 + (y-1/2)^2 = \frac{17}{4}$
Hence, the curve is a circle with the center $(2, 1/2)$ and radius $\sqrt{17}/2$.

(b)(5pts) Find the equation of the tangent line to the curve at the point $\theta = \pi/4$.

Solution.
For an equation of a tangent line on the plane we need a slope and a point. Observe that we can interpret the curve given in polar coordinates as a parametric curve with the parameter θ:

$$(x, y) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$$

The slope is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r(\theta)\sin\theta)}{\frac{d}{d\theta}(r(\theta)\cos\theta)} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

Using the equaion $r = 4\cos\theta + \sin\theta$ which defines the curve we compute

$$r'(\theta) = -4\sin\theta + \cos\theta.$$

Hence,

$$r(\pi/4) = 5/\sqrt{2}; \quad r'(\pi/4) = -3/\sqrt{2}.$$

Plugging these values together with $\theta = \pi/4$ into the formula for dy/dx above, we get

$$\frac{dy}{dx} = \frac{-\frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}\frac{1}{\sqrt{2}}}{-\frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}\frac{1}{\sqrt{2}}} = -\frac{1}{4}.$$

So, we have found the slope. Do not stop here! To find a point on the line, we plug in $\theta = \pi/4$ into the equations for x, y:

$$x = r(\pi/4)\cos(\pi/4) = \frac{5}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{5}{2}; \quad y = r(\pi/4)\sin(\pi/4) = \frac{5}{2}.$$

Hence, the point is $(\frac{5}{2}, \frac{5}{2})$. The equation of the line in the point-slope form is

$$y - \frac{5}{2} = -\frac{1}{4}(x - \frac{5}{2})$$

2. (12pts) Consider the parametric curve given by the vector function

$$\vec{r}(t) = (t, t^2, t^3)$$

(this curve is called a *twisted cubic*).

(a)(4pts) Find the equation of the normal plane at the point when $t = 1$.

Hint. The normal plane is the plane perpendicular to the tangent line.

Solution. $\vec{r}'(t) = (1, 2t, 3t^2)$;
$\vec{r}'(1) = (1, 2, 3)$;
and this is the normal vector to the normal plane in question. The point on the curve at the time $t = 1$ has coordinates $(1, 1, 1)$. Hence, the equation is

$$(x - 1) + 2(y - 1) + 3(z - 1) = 0$$

or $x + 2y + 3z = 6$.

(b)(4pts) Find the equation of the normal plane at the point $(-1, 1, -1)$.

Solution. First, we find t corresponding to the given point. Since $(t, t^2, t^3) = (-1, 1, -1)$, we conclude $t = -1$. Now proceed as in part (a).

$\vec{r}'(-1) = (1, -2, 3)$;
this is the normal vector to the normal plane in question. The point is $(-1, 1, -1)$. Hence, the equation of the plane is

$$(x + 1) - 2(y - 1) + 3(z + 1) = 0$$

or $x - 2y + 3z = -6$

(c)(4pts) Find the parametric equations of the line of intersection of the planes from (a) and (b).

Solution. $\vec{v} = \vec{r}'(1) \times \vec{r}'(-1) = (1, 2, 3) \times (1, -2, 3) = (12, 0, -4)$ is the direction vector for the intersection line; we can cancel 3 and take $(3, 0, -1)$. To find one point on the intersection, set $z = 0$ and solve
$x + 2y = 6$
$x - 2y = -6$
We get $x = 0, y = 3$. Hence a point on the intersection is $(0, 3, 0)$. The parametric equations are $x = 3t, y = 3, z = -t$.

3. (12pts) Consider the surface defined by the equation $f(x, y) = x^2y + y^3 + x$.

(a)(6pts) Find the tangent plane to the surface at the point $(-2, 1, 3)$.

Solution. $f_x(x, y) = 2xy + 1$; $f_y(x, y) = x^2 + 3y^2$;
at the point $(-2, 1)$, we have $f_x = -3$, $f_y = 7$. Hence, the equation is

$$z - 3 = -3(x + 2) + 7(y - 1)$$

(b)(6pts) Find all second partial derivatives of $f(x, y)$.

Solution. $f_{xx} = 2y$
$f_{xy} = 2x$
$f_{yy} = 6y$

4. (14 pts) (a)(5pts) Find the velocity and position vectors of a particle that has the acceleration vector

$$\vec{a}(t) = \langle 2, \cos t, \sin t \rangle,$$

the initial velocity $\vec{v}(0) = \langle 0, 0, -1 \rangle$ and the initial position $\vec{r}(0) = \langle 1, 1, 0 \rangle$.

Solution. Integrate $\vec{a}(t)$ to get the velocity.
$\vec{v}(t) = \langle 2t + c_1, \sin t + c_2, -\cos t + c_3 \rangle$. Since $\vec{v}(0) = \langle 0, 0, -1 \rangle$, we get $\langle c_1, c_2, -1 + c_3 \rangle = \langle 0, 0, -1 \rangle$. Hence, $c_1 = c_2 = c_3 = 0$. We obtain

$$\vec{v}(t) = \langle 2t, \sin t, -\cos t \rangle.$$

Integrating again, we get $\vec{r}(t) = \langle t^2 + d_1, -\cos t + d_2, -\sin t + d_3 \rangle$. Using the initial position $\vec{r}(0) = \langle 1, 1, 0 \rangle$, we compute the constants $d_1 = 1, d_2 = 2, d_3 = 0$. Hence,

$$\vec{r}(t) = \langle t^2 + 1, -\cos t + 2, -\sin t \rangle.$$

(b)(1pt) Find the position vector at the time $t = 1$.

Solution. $\vec{r}(1) = \langle 2, -\cos 1 + 2, -\sin 1 \rangle$.

Answer the following two questions in any order. Simplify your answers as much as possible.

(c)(4pts) Find the curvature at $t = 1$.

(d)(4pts) Find the length of the projection of the acceleration vector at $t = 1$ on the unit Normal vector at $t = 1$.

Solution. The length of the projection of the acceleration vector on the unit Normal vector is the "normal component" of the acceleration, denoted a_N or $||\vec{a}_N||$. In particular, the answers to both c) and d) are SCALARS.

We shall use the formula

$$a_N = \frac{||\vec{v} \times \vec{a}||}{||\vec{v}||}$$

to solve d), and then

$$\kappa = \frac{a_N}{||\vec{v}||^2}$$

to quickly get c) from d).

(Remark: One can also go the opposite direction, and first find κ using the formula $\kappa = \frac{||\vec{v} \times \vec{a}||}{||\vec{v}||^3}$ and then compute $a_N = \kappa ||\vec{v}||^2$.

Either way, we need to find the cross product $\vec{a} \times \vec{v}$, and then compute its magnitude. Plugging in $t = 1$ into the formula we found in a), we get

$$\vec{v}(1) = \langle 2, \sin 1, -\cos 1 \rangle$$

and

$$\vec{a}(1) = \langle 2, \cos 1, \sin 1\rangle$$

Hence,

$$\vec{v}(1) \times \vec{a}(1) = \langle \sin^2(1) + \cos^2(1), -2\sin 1 - 2\cos 1, 2\cos 1 - 2\sin 1\rangle =$$

$$\langle 1, -2\sin 1 - 2\cos 1, 2\cos 1 - 2\sin 1\rangle$$

Thus,

$$||\vec{v}(1) \times \vec{a}(1)|| = \sqrt{1 + (-2\sin 1 - 2\cos 1)^2 + (2\cos 1 - 2\sin 1)^2} =$$

(This may look rather cumbersome but don't get discouraged: a little bit of FOILing and simplifying reveals a nice answer:)

$$\sqrt{1 + 4\sin^2(1) + 8\sin 1\cos 1 + 4\cos^2(1) + 4\cos^2(1) - 8\sin 1\cos 1 + 4\sin^2(1)} =$$

$$\sqrt{1 + 8\sin^2(1) + 8\cos^2(1)} = \sqrt{9} = 3$$

Next, we find $||\vec{v}(1)|| = \sqrt{2^2 + \sin^2(1) + \cos^2(1)} = \sqrt{5}$. Finally,

$$a_N = \frac{||\vec{v} \times \vec{a}||}{||\vec{v}||} = \frac{3}{\sqrt{5}}$$

and

$$\kappa = \frac{a_N}{||\vec{v}||^2} = \frac{3}{5\sqrt{5}}$$

5. (3pts) (*Bonus, full credit only).* Show that if a particle moves with the constant speed, then the velocity and acceleration vectors are orthogonal.