MATH **126** Quiz **Nov. 28** **Name:**

No books, notes or graphing calcuators. Please turn off your cell phones. Show ALL your work.

(10) 1. A rectangular box without a lid is to be made from 100m^2 of cardboard.

(a) What is the maximal volume of such a box?

W e give two solutions. The first uses implicit differentiation, and the second solves explicitly for one of the variables.

Solution I. Let x be the length, z be the width and y be the height of the box (note the unusual choice of variables). Since we are looking for the maximal volume, we may assume $x, y, z > 0$

The volume is

$$V = xyz$$

and the total surface area is

$$100 = xz + 2xy + 2zy \qquad (*)$$

We treat $z = z(x, y)$ as a function of x, y without solving for it and use implicit differentiation. The volume $V = xyz$ becomes a function of x, y as well. To maximize the volume V, we have to find the critical points. For this, we have to solve $V_x = V_y = 0$.

Find the partial derivatives of V and set them equal to 0

$$V_x = yz + xyz_x = 0, \quad V_y = xz + xyz_y = 0$$

Simplify

$$z + xz_x = 0$$

$$z + yz_y = 0$$

Subtract

$$xz_x = yz_y \qquad (**)$$

Next we differentiate the total surface area.

Take $\frac{\delta}{\delta x}$ of (*)

$$0 = z + xz_x + 2y + 2yz_x$$

Solve for z_x

$$z_x = -\frac{z + 2y}{x + 2y}$$

Now take $\frac{\delta}{\delta y}$ of (*)

$$0 = xz_y + 2x + 2z + 2yz_y$$

Solve for z_y

$$z_y = -\frac{2x + 2z}{x + 2y}$$

Plug in the formulas for z_x, z_y into (**)

$$x\frac{z + 2y}{x + 2y} = y\frac{2x + 2z}{x + 2y}$$

Simplify

$$xz + 2xy = 2xy + 2zy$$

Simplify further

$$x = 2y$$

Since x and z are "symmetric" variables in the problem, we must also have

$$z = 2y$$

Plug $x = z = 2y$ into the equation $100 = xz + 2yx + 2yz$. We get $100 = 4y^2 + 4y^2 + 4y^2$. Hence, $y = \sqrt{\frac{100}{12}} = \frac{5}{\sqrt{3}}$. The volume $V = xyz = 4y^3 = \frac{500}{3\sqrt{3}}$.

Solution II. Let x and y be the dimensions of the bottom of the box and let z be its height. Then the volume of the box is $V(x, y, z) = xyz$ and we want to maximize this function. The surface area of the box is xy for the bottom plus $2xz$ and $2yz$ for the sides. Thus we must have $100 = xy + 2xz + 2yz$. Solving this relation for z gives us $z = \dfrac{100 - xy}{2x + 2y}$. If we plug it into the formula for V, we obtain $V(x, y) = \dfrac{100xy - x^2y^2}{2x + 2y}$. In order to find critical points of this function, we compute its partial derivatives:

$$\begin{aligned} V_x &= \frac{(100y - 2xy^2)(2x + 2y) - (100xy - x^2y^2)2}{(2x + 2y)^2} = \frac{(100y - 2xy^2)(x + y) - (100xy - x^2y^2)}{2(x + y)^2} \\ &= \frac{100xy + 100y^2 - 2x^2y^2 - 2xy^3 - 100xy + x^2y^2}{2(x + y)^2} = \frac{100y^2 - x^2y^2 - 2xy^3}{2(x + y)^2} \\ &= y^2 \frac{100 - x^2 - 2xy}{2(x + y)^2}. \end{aligned}$$

By symmetry,

$$V_y = x^2 \frac{100 - y^2 - 2xy}{2(x + y)^2}.$$

Since we are looking for a point of maximum, we may assume that $x \neq 0$ and $y \neq 0$. Then $V_x = V_y = 0$ is equivalent to

$$100 - x^2 - 2xy = 0 = 100 - y^2 - 2xy.$$

We see that $x^2 = y^2$, thus $x = y$, since both must be positive. Then $100 - x^2 - 2x^2 = 100 - 3x^2 = 0$ and $x = \sqrt{\frac{100}{3}}$. Since we have only one critical point and we know that there is the largest box, this point must be the point of maximum and when $x = y = \sqrt{\frac{100}{3}}$, we have

$$V = \frac{100xy - x^2y^2}{2x + 2y} = \frac{100x^2 - x^4}{4x} = x\frac{100 - x^2}{4} = \sqrt{\frac{100}{3}}\left(25 - \frac{25}{3}\right) = \frac{10}{\sqrt{3}}\frac{50}{3} = \frac{500}{3\sqrt{3}} \approx 96.$$

(b) Assume that the box must be at least 10m tall. What is the maximal volume the box in this case?

Solution. At the critical point found above we have $z = \dfrac{100 - xy}{2x + 2y} = \dfrac{100 - x^2}{4x} = \dfrac{5}{\sqrt{3}} < 10$. Thus there are no critical points inside the region $z \geqslant 10$ and the maximum value is attained on the boundary, i.e. we must have $z = 10$. Then we need to maximize $V(x, y) = 10xy$ with the condition $100 = xy + 20x + 20y$. Solving this equation for y gives $y = \dfrac{100 - 20x}{x + 20}$. Thus we need to maximize $V(x) = 100\dfrac{10x - 2x^2}{x + 20}$. We compute

$$V'(x) = 100\frac{(10 - 4x)(x + 20) - (10x - 2x^2)}{(x + 20)^2} = 100\frac{10x - 4x^2 + 200 - 80x - 10x + 2x^2}{(x + 20)^2} = 100\frac{200 - 80x - 2x^2}{(x + 20)^2},$$

so $V'(x) = 0$ if $100 - 40x - x^2 = 0$, or $x^2 + 40x - 100 = 0$. The positive solution of this equation is $x = \dfrac{-40 + \sqrt{1600 + 400}}{2} = -20 + 10\sqrt{5}$. By symmetry, if we were looking for the critical value of y, we would get the same number. Therefore, the maximum volume is $V = 10(10\sqrt{5} - 20)^2 = 10(500 + 400 - 400\sqrt{5}) = 1000(9 - 4\sqrt{5}) \approx 56$.