

1.)

$$\begin{aligned} & \int \frac{dt}{\sqrt{t^2 - 6t + 13}} \\ &= \int \frac{dt}{\sqrt{(t-3)^2 + 4}} \end{aligned}$$

Let $t - 3 = 2 \tan x$

The integral becomes:

$$\begin{aligned} & \int \frac{2 \sec^2 x dx}{\sqrt{4 \tan^2 x + 4}} \\ &= \int \sec x dx \\ &= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{\sqrt{4 + (t-3)^2} + t-3}{\sqrt{4 + (t-3)^2} - t+3} \right| + C \end{aligned}$$

2.) Method 1 (partial fractions):

$$\begin{aligned} & \int \frac{dt}{t^2 - 6t + 8} \\ &= \int \frac{dt}{(t-2)(t-4)} \\ &= \int \left(\frac{-1/2}{t-2} + \frac{1/2}{t-4} \right) dt \\ &= -\frac{1}{2} \ln |t-2| + \frac{1}{2} \ln |t-4| + C \\ &= \frac{1}{2} \ln \left| \frac{t-4}{t-2} \right| + C \end{aligned}$$

Method 2 (trig substitution)

$$\begin{aligned}& \int \frac{dt}{t^2 - 6t + 8} \\&= \int \frac{dt}{(t-3)^2 - 1}\end{aligned}$$

Let $t - 3 = \sin x$, then the integral becomes:

$$\begin{aligned}& \int \frac{\cos x}{\sin^2 x - 1} dx \\&= \int \frac{\cos x}{-\cos^2 x} dx \\&= - \int \frac{1}{\cos x} dx \\&= - \int \sec x dx\end{aligned}$$

Use formula given in question (1):

$$\begin{aligned}& - \int \sec x dx \\&= -\frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C \\&= -\frac{1}{2} \ln \left| \frac{1 + (t-3)}{1 - (t-3)} \right| + C \\&= \frac{1}{2} \ln \left| \frac{t-4}{t-2} \right| + C\end{aligned}$$

Bonus problem:

$$\begin{aligned}& \int \frac{dx}{1+\cos x} \\&= \int \frac{dx}{1+(2\cos^2 \frac{x}{2}-1)} \\&= \frac{1}{2} \int \sec^2 \frac{x}{2} dx\end{aligned}$$

Let

$$\begin{aligned}u &= \tan \frac{x}{2} \\ \Rightarrow du &= \frac{1}{2} \sec^2 \frac{x}{2} dx\end{aligned}$$

Hence the integral becomes:

$$\begin{aligned}\int du \\&= u + C \\&= \tan \frac{x}{2} + C\end{aligned}$$