MATH **125** Quiz **Oct. 30** **Name:**

No books, notes or graphing calculators. Turn off your cell phones. Good luck!

Evaluate the following integrals:

(4) 1. $\int x^2 \sin x \; dx$

You integrate this by doing an integration by parts twice. First use

$$u = x^2, \quad dv = \sin(x)\; dx,$$

so

$$du = 2x\; dx, \quad v = -\cos(x),$$

and integration by parts gives

$$\int x^2 \sin(x)\; dx = -x^2\cos(x) + \int 2x\cos(x)\; dx = -x^2\cos(x) + 2\int x\cos(x)\; dx. \qquad (\star)$$

To evaluate $\int x\cos(x)\; dx$, let $u = x$ and $dv = \cos(x)\; dx$, so

$$du = dx, \quad v = \sin(x),$$

and

$$\int x\cos(x)\; dx = x\sin(x) - \int \sin(x) = x\sin(x) + \cos(x) + C.$$

Substituting that integral into $\star$ we finally have

$$\int x^2\sin(x)\; dx = -x^2\cos(x) + 2\int x\cos(x)\; dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

(note that the constant C in the above equation is different from the constant C before because it's been multiplied by 2, but since the constant is unspecified we just absorb the 2 into the constant).

(3) 2. $\int_0^{\frac{\pi}{2}} \sin^2 x \; dx$

There are (at least) 3 different ways of doing this problem. Two are use integration by parts and the third uses a trigonometric identity to find the anti-derivative.

(1) The first way is the hardest. Let $u = \sin^2(x)$ and $dv = dx$. Then

$$du = 2\sin(x)\cos(x)\; dx = \sin(2x)\; dx, \quad v = x,$$

so

$$\int \sin^2(x)\; dx = x\sin^2(x) - \int x\sin(2x)\; dx. \qquad (\star)$$

We can evaluate the integral on the left-hand side by letting $u = x$ and $dv = \sin(2x)\; dx$, so $du = dx$ and $v = -\cos(2x)/2$, so

$$\int x\sin(2x)\; dx = -\frac{x\cos(2x)}{2} + \int \frac{\cos(2x)}{2}\; dx = -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + C. \qquad (\clubsuit)$$

Substituting $\clubsuit$ into $\star$ we have the (nearly) final answer

$$\int \sin^2(x)\; dx = x\sin^2(x) + \frac{x\cos(2x)}{2} - \frac{\sin(2x)}{4} + C.$$

To show this indefinite integral is the same as the other two indefinite integrals we shall derive, recall that $\cos(2x) = \cos^2(x) - \sin^2(x)$ and note that

$$\begin{aligned} x\sin^2(x) + \frac{x\cos(2x)}{2} &= x\left(\sin^2(x) + \frac{\cos(2x)}{2}\right) \\ &= x\left(\sin^2(x) + \frac{\cos^2(x) - \sin^2(x)}{2}\right) \\ &= x\left(\frac{\cos^2(x) + \sin^2(x)}{2}\right) \\ &= x\left(\frac{1}{2}\right) \\ &= \frac{x}{2}. \end{aligned}$$

So our (final) final answer is

$$\int \sin^2(x)\ dx = x\sin^2(x) + \frac{x\cos(2x)}{2} - \frac{\sin(2x)}{4} + C = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

(2) The second way to evaluate the integral uses $u = \sin(x)$ and $dv = \sin(x)\ dx$. This method is still harder than just using a trigonometric identity, but it is a somewhat common trick for integration by parts. By our choice of u and dv we have $du = \cos(x)$ and $v = -\cos(x)\ dx$, so

$$\int \sin^2(x)\ dx = -\sin(x)\cos(x) + \int \cos^2(x)\ dx.$$

Now we use the fact that $\cos^2(x) = 1 - \sin^2(x)$ to get

$$\int \sin^2(x)\ dx = -\sin(x)\cos(x) + \int 1\ dx - \int \sin^2(x)\ dx = -\frac{\sin(2x)}{2} + x + C + \int \sin^2(x)\ dx,$$

so adding $\int \sin^2(x)\ dx$ to both sides and dividing by 2 we get

$$\int \sin^2(x)\ dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

(3) The third and final method is the easiest. We just use the trig identity

$$\sin^2(x) = \frac{1 - \cos(2x)}{2},$$

so

$$\begin{aligned} \int \sin^2(x)\ dx &= \int \frac{1 - \cos(2x)}{2}\ dx \\ &= \int \frac{1}{2}\ dx - \int \frac{\cos(2x)}{2}\ dx \\ &= \frac{x}{2} - \frac{\sin(2x)}{4} + C. \end{aligned}$$

Regardless which method is used, something equivalent to the anti-derivative $(x/2) - (\sin(2x)/4)$ is obtained, so the definite integral is

$$\begin{aligned} \int_0^{\pi/2} \sin^2(x)\ dx &= \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\pi/2} \\ &= \left(\frac{\pi/2}{2} - \frac{\sin(\pi)}{4}\right) - \left(\frac{0}{2} - \frac{\sin(0)}{4}\right) \\ &= \frac{\pi}{4}, \end{aligned}$$

which is the solution to the entire problem.

(3) 3. $\int \tan x \sec^2 x \, dx$

There are some ways of evaluating this integral using integration by parts, but they are so much more labor intensive for no reason that they will not be presented here. If you have questions about evaluting this integral using integration by parts ask the professor or TA during office hours.

(1) This integral can be evaluated pretty easily using substitution. You could use $u = \tan(x)$, so $du = \sec^2(x)\, dx$, and the integral becomes

$$\int \tan(x) \left(\sec^2(x)\, dx\right) = \int u \, du = \frac{u^2}{2} + C = \frac{\tan^2(x)}{2} + C.$$

(2) Alternatively, you could use $u = \sec(x)$ so $du = \sec(x)\tan(x)\, dx$ so the integral becomes

$$\int \sec(x) \left(\sec(x)\tan(x)\, dx\right) = \int u \, du = \frac{u^2}{2} + C = \frac{\sec^2(x)}{2} + C.$$

These two solutions agree because $\sec^2(x) = \tan^2(x) + 1$, so the two answers agree up to a constant (namely 1/2), which is all indefinite integrals need to agree up to in order to be the same.

4. [1 Bonus point.] Velocity of a car satisfies the equation $v(t) = \sin(9t)\cos(8t)$ m/s. Show that after traveling for 2π seconds the car comes back to its starting point.

The problem wants you to show the position $s(t) = \int_0^t v(t)\, dt$ is at your starting point. Assuming the starting point is 0, the problem is to show

$$\int_0^{2\pi} v(t)\, dt = 0.$$

There are two different methods of solving this problem. The first is to use trigonometric identities to simplify $\sin(9t)\cos(8t)$ to evaluate the resulting integral, and the second is to realize that $v(t)$ is an odd function about $x = \pi$, so the required integral must vanish.

(1) For the first method, use the identity

$$\sin(9t)\cos(8t) = \frac{1}{2}\left(\sin((9-8)t) + \sin((9+8)t)\right) = \frac{1}{2}\left(\sin(t) + \sin(17t)\right),$$

so

$$\begin{aligned}
\int_0^{2\pi} v(t)\, dt &= \int_0^{2\pi} \frac{1}{2}\left(\sin(t) + \sin(17t)\right)\, dt \\
&= \frac{1}{2}\int_0^{2\pi} \sin(t) + \sin(17t)\, dt \\
&= \frac{1}{2}\left[-\cos(t) - \frac{\cos(17t)}{17}\right]_0^{2\pi} \\
&= \frac{1}{2}\left(\left(-\cos(0) - \frac{\cos(0)}{17}\right) - \left(-\cos(2\pi) - \frac{\cos(34\pi)}{17}\right)\right) \\
&= \frac{1}{2}\left(\left(-1 - \frac{1}{17}\right) - \left(-1 - \frac{1}{17}\right)\right) \\
&= \frac{1}{2}\left(-\frac{18}{17} + \frac{18}{17}\right) \\
&= 0.
\end{aligned}$$

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(2) For the slightly easier method note that if you perform the change of variables $u = t - \pi$ the integral becomes

$$\int_{-\pi}^{\pi} \sin(9(u+\pi))\cos(8(u+\pi))\ du,$$

which is zero because $\sin(9(u+\pi))\cos(8(u+\pi))$ is odd. To see $\sin(9(u+\pi))\cos(8(u+\pi))$ is odd, note that if we replace u by $-u$ then, since sin and cos are 2π-periodic (i.e. $\sin(x+2\pi) = \sin(x)$ and $\cos(x+2\pi) = \cos(x)$), then

$$\begin{aligned}
\sin(9(-u+\pi))\cos(8(-u+\pi)) &= \sin(9(-u+\pi) - 18\pi)\cos(8(-u+\pi) - 16\pi) \\
&= \sin(-9(u+\pi))\cos(-8(u+\pi)) \\
&= -\sin(9(u+\pi))\cos(8(u+\pi)),
\end{aligned}$$

which shows $\sin(9(u+\pi))\cos(8(u+\pi))$ is odd because replacing u with $-u$ gives $-\sin(9(u+\pi))\cos(8(u+\pi))$.