Some answers to Midterm II practice problems

Math 125, Sections C&D

November, 2007

1. Problems 5, 6 and 7 from week 6 homework (lots of useful practice!)
2. Evaluate the following integrals:
   (a) $\int x\cos^2(x)\ dx = \frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8} + C$
   (b) $\int x\sec^2(x)\tan(x)\ dx = \frac{x\sec^2 x - \tan x}{2} + C$
   (c) $\int \frac{e^x}{e^{3x}+3e^{2x}+3e^x}\ dx = \frac{x}{3} + \frac{1}{6}\ln|e^{2x}+3e^x+3| + \frac{1}{\sqrt{3}}\arctan(\frac{2e^x+3}{\sqrt{3}}) + C$
   (d) $\int \sec^2(x)\sqrt{\tan^2(x)-4}\ dx = 2\ln|\tan x - \sqrt{\tan^2 x - 4}| + \frac{\tan x}{2}\sqrt{\tan^2 x - 4} + C$
   Hint: First, substitute $u = \tan x$. Then use trig substitution. Then either rewite the resulting integral in terms of sin and cos and apply the strategy for solving trig integrals or use integration by parts. If you choose the second approach, then the trig identity $\sec^2\theta = \tan^2\theta + 1$ may be useful. OTHER APPROACHES ARE POSSIBLE.
   (e) $\int \frac{\sqrt{\tan^2(x)+9}}{\sin^2(x)}\ dx = -\frac{\sqrt{\tan^2 x + 9}}{\tan x} + \ln|\sqrt{\tan^2 x + 9} + \tan x| + C$
   Hint: use the same strategy as for d).

   Caution: The last two questions have many different forms of correct answers. Please come to office hours if you would like for someone to check your work.
3. Use trapezoidal rule with $n = 4$ to estimate the area under the graph of $y = \sqrt{2+x^4}$ on the interval $[0, 2]$ (First set up the integral, then estimate. Leave your answer in the sum form. Do not simplify or evaluate.)

   Answer. $T_4 = \frac{1}{4}(\sqrt{2} + 2\sqrt{2+(\frac{1}{2})^4} + 2\sqrt{2+1} + 2\sqrt{2+(\frac{3}{2})^4} + \sqrt{2+2^4}) = \frac{1}{4}(\sqrt{2} + \frac{\sqrt{33}}{2} + 2\sqrt{3} + \frac{\sqrt{113}}{2} + 3\sqrt{2})$.
4. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$.
   (a) Sketch $\mathcal{R}$.

   For b) and c) indicate clearly whether you are using shells or washers. Sketch a typical rectangle to be rotated to obtain either a shell or a washer, depending on the method you use.

(b) Compute the volume obtained by rotating $\mathcal{R}$ around the x-axis.

Answer. Using washers, the integral for the volume is $\pi \int_0^1 (2x - x^2)^2 - x^6 \, dx = \pi \frac{41}{105}$.

(c) Compute the volume obtained by rotating $\mathcal{R}$ around the line $x = -1$.

Answer. Here, we use shells. Our variable is again x. The surface area of a typical shell is $2\pi r h = 2\pi(x+1)(2x - x^2 - x^3)$. Hence, the volume is

$$2\pi \int_0^1 (x+1)(2x - x^2 - x^3) \, dx = \pi \frac{19}{15}$$

5. Sketch the region in the first quadrant bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, the line $x = 2y$, and the x-axis. Find the average height of the region. (You can give your answer either as a decimal or in exact form.)

Sketch of solution. First, find the intersection point of the ellipse and the line. For this, plun i $x = 2y$ into the equation of the ellipse and solve. You should get $x = \frac{12}{5}$, $y = \frac{6}{5}$.

Divide the region into two parts by a vertical line at $x = \frac{12}{5}$. The left part is a triangle with the sides $\frac{12}{5}$ and $\frac{6}{5}$, and the area $\frac{1}{2}(\frac{12}{5} \cdot \frac{6}{5}) = \frac{36}{25}$. To find the area of the right-hand side of the region we set up the integral

$$Area = \int_{12/5}^{3} 2\sqrt{1 - \frac{x^2}{9}} \, dx = \frac{2}{3} \int_{12/5}^{3} \sqrt{9 - x^2} \, dx = \frac{2}{3}\left(\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2} \arcsin \frac{x}{3}\right)\Bigg|_{12/5}^{3} =$$

$$\frac{3\pi}{2} - \frac{36}{25} - 3 \arcsin \frac{4}{5}$$

Hence, the total area is $\frac{36}{25} + \frac{3\pi}{2} - \frac{36}{25} - 3 \arcsin \frac{4}{5} = \frac{3\pi}{2} - 3 \arcsin \frac{4}{5}$. To get the average height we divide by the length of the base which is 3, and get $\boxed{\frac{\pi}{2} - \arcsin \frac{4}{5} = 0.64}$

6. A spring has a natural length of 20cm. If a 25N force is required to keep it stretched to a length of 30cm, how much work is required to stretch it from 20cm to 25cm?

We did this problem in class, it is also #8 from 6.4.

7. A 20-foot rope hangs over the edge of a cliff. It rained earlier so the rope is wet, and since the water tends to seep downwards the bottom of the rope is heavier than the top. Suppose that the weight density of the wet rope at the distance y feet from the top is $1 + \frac{3}{80}y$ lb/ft. Calculate the work needed to pull the rope up to the top.

Answer. $W = \int_0^{20} (1 + \frac{3}{80}y) y \, dy = 300$ ft-lb.

8. A tank full of water has the shape of a hemisphere. The diameter of the tank is 10m. Set up (but do not evaluate) the integral for the work required to pump all of the water out of the tank (through the top).

   Answer. $W = \int_0^5 (\pi(25-y^2)1000\cdot 9.8)y\, dy = 9800\pi \int_0^5 25y - y^3\, dy = 9800\pi(\frac{25}{2}y^2 - \frac{y^4}{4})\Big|_0^5 = 9800 \cdot \frac{625}{4}\pi = 1531250$ J.

9. A bag of sand originally weighs 160 lbs. It is lifted at a constant rate of 4 ft/min. The sand leaks out of the bag at a constant rate so that when it has been lifted 20 ft only half of the sand is left. How much work is done lifting the bag 20 ft?

   Answer. $W = \int_0^{20} (160 - 4y)\, dy = 2400$ ft-lb.

10. Consider the region in the first quadrant bounded by $y = x^{\frac{3}{2}}$, $x = 0$, and $y = 8$. This region is revolved around the y-axis to create a three dimensional solid. Suppose we have a tank with the shape of that solid, oriented so that the y-axis is perpendicular to the ground, the origin is at the bottom of the tank, and units are in meters (so the tank is 8m tall). If the tank is filled with a liqiud with mass density 2300kg/m^3, how much work is required to pump all of the liquid to the top of the tank?

    Answer. Solve for x: $x = y^{2/3}$.
    The area at the height y is $\pi x^2 = \pi y^{4/3}$.
    $m = Area \times density = 2300\pi y^{4/3}$.
    $F = mg = 2300\pi y^{4/3} \cdot 9.8 = 22540\pi y^{4/3}$
    $W = \int_0^8 22540\pi y^{4/3}(8-y)\, dy = 22540\pi \int_0^8 8y^{4/3} - y^{7/3}\, dy = 22540\pi(\frac{24}{7}y^{7/3} - \frac{3}{10}y^{10/3})\Big|_0^8 = 22540\pi(\frac{24}{7}2^7 - \frac{3}{10}2^{10}) \simeq 3 \cdot 10^6\pi$ J.

11. The line $y = 3x$, for $0 \le x \le 1$, it rotated around the y-axis to form a cone (units are in feet). The cone is filled with melted ice cream, which weighs 59.2 lb/ft^3. How much work does it take to pump all of the ice cream up to the height $y = 10$.

    Answer. Put the origin at the tip of the tank. Then radius at the height y is $y/3$ (use similar triangles). The area at the height y is $\pi(\frac{y}{3})^2$. Hence, $dF = Area \times density\ dy = \pi\frac{y^2}{9} \times 59.2\ dy = 6.57\pi y^2 dy$. We get

    $$W = \int_0^3 6.57\pi y^2(10-y)dy = 455.5\pi \text{ ft-lb}$$

12. (more difficult) Newton's law of Universal Gravitation implies that an object in Earth's gravitational field located r km from the center of the Earth experiences gravitational acceleration $g(r) = \frac{c}{r^2}$ m/s^2. Assume that the gravitational acceleration on the surface of the Earth is 9.8 m/s^2 and the radius of the Earth is equal to 6,500km. The

"geostationary" orbit is about 42,000km from the center of the Earth. A rocket takes a satellite to the orbit. The initial mass of the rocket and the satellite is 5,000kg. The mass decreases as the fuel burns. Assume that the mass is a linear function of the distance from the center of the Earth. The final mass of the satellite at the geostationary orbit is 1000kg. What is the total amount of work needed to place satellite in its orbit? Ignore the gravitational attraction of other celestial bodies such as the Sun and the Moon.